

Games and Economic Theory  
Selected Contributions in Honor of Robert J. Aumann  
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INTRODUCTION

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**Robert J. Aumann: An Overview of His Work**

*Robert J. Aumann* is an eminent scientist. He is one of the greatest thinkers on all aspects of *rationality in decision-making*. Aumann has played an essential and indispensable role in shaping *game theory* and much of *economic theory*, to become the great success it is today. He promotes a unified view of the very wide domain of rational behavior, a domain that encompasses areas of many apparently disparate disciplines, like economics, political science, biology, psychology, mathematics, philosophy, computer science, law, and statistics. Aumann's research is characterized by an unusual combination of breadth and depth. His scientific contributions are path-breaking, innovative, comprehensive, and rigorous – from the discovery and formalization of the basic concepts and principles, through the development of the appropriate tools and methods for their study, to their application in the analysis of various specific issues. Some of his contributions require very deep and complex technical analysis; others are (as he says at times) “embarrassingly trivial” mathematically, but very profound conceptually. They are all insightful and thought-provoking, and go into the roots and heart of the central issues. It is science at its best.

Half a century ago, the collaboration of the mathematician John von Neumann and the economist Oskar Morgenstern resulted in the 1944 publication of their book *Theory of Games and Economic Behavior*.<sup>1</sup> This is the starting point of the scientific discipline called “game theory.”

What is game theory? A better name, suggested by Aumann [55, p. 460],<sup>2</sup> is perhaps “interactive decision theory”. The object of study is the interaction of decision-makers (“players”) whose decisions affect each other. The analysis is from a “rational” viewpoint; that is, each participant would like to obtain those outcomes he prefers most. When there is only one player, this usually leads to a well-defined optimization problem. In contrast, in the multi-person setup of game theory, the preference ranking of a player over the outcomes

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<sup>1</sup>J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior* (Princeton: Princeton University Press, 1944).

<sup>2</sup>Bracketed citations refer to publications by Aumann, as listed in Appendix B.

does not translate into a ranking over his possible decisions – the outcome also depends on the decisions of the other participants.

Game theory deals, first, with the fundamental issue of defining the concept of “optimal rational decision” in interactive situations. Second, it analyzes these “solution concepts” both in the general setup as well as in particular models. Game theory always strives to develop general and universal approaches, rather than using an ad-hoc analysis that deals with each specific issue separately. To quote Aumann,

Game theory is a sort of umbrella or “unified field” theory for the rational side of social science, where “social” is interpreted broadly, to include human as well as non-human players (computers, animals, plants). Unlike other approaches to disciplines like economics and political science, game theory does not use different, ad-hoc constructs to deal with various specific issues, such as perfect competition, monopoly, oligopoly, international trade, taxation, voting, deterrence, and so on. Rather, it develops methodologies that apply in principle to all interactive situations, then sees where these methodologies lead in each specific application. [55, p. 460]

Robert J. Aumann has contributed, probably no less than anyone else, to the great development of game theory in the past decades, and to the establishment of its central role in economic theory. He has influenced and shaped the field through his pioneering work. There is hardly an area of game theory today where his footsteps are not readily apparent. Most of Aumann’s research is intimately connected to central issues in economic theory; on one hand, these issues provided the motivation and impetus for his work; on the other hand, his results produced novel insights and understandings in economics. In addition to his own pioneering work, Aumann’s indirect impact is no less important – through his many students, collaborators, and colleagues. He inspired them, excited them with his vision, and led them to further important results.

In the following four sections we will survey Aumann’s main contributions.

## 1. Perfectly Competitive Economies

A *perfectly competitive* economic model is meant to describe a situation where there are many participants, and such that the influence of each one individually is negligible. The state of the economy is thus insensitive to the actions of any single agent; only the aggregate behavior matters. For instance, in a pure exchange economy in which the initial endowment of each trader is very small relative to the whole, the quantities of goods traded by any one agent cannot essentially affect the total supply and demand.

The first question is: What is the correct way of modeling perfect competition? Aumann introduced the model of economies with a continuum of participants, as the appropriate model where each individual is indeed insignificant:

Indeed, the influence of an individual participant on the economy cannot be mathematically negligible, as long as there are only finitely many participants. Thus a *mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants*. We submit that the most natural model for this purpose contains a continuum of participants, similar to the continuum of points on a line or the continuum of particles in a fluid. [16, p. 39]

The introduction of the “continuum” idea in economic theory has been indispensable to the advancement of this discipline. In the same way as in most of the natural sciences, it enables a precise and rigorous analysis, which otherwise would have been very hard or even impossible.

the continuum can be considered an approximation to the “true” situation in which there is a large but finite number of particles (or traders, or strategies, or possible prices). The purpose of adopting the continuous approximation is to make available the powerful and elegant methods of the branch of mathematics called “analysis”, in a situation where treatment by finite methods would be much more difficult or even hopeless (think of trying to do fluid mechanics by solving n-body problems for large n). [16, p. 41]

Once the basic model is specified, the next question is: What does perfect competition lead to? The classical economic approach is that there are prices for all goods, which every agent takes as given (he is, after all, insignificant, so his decision cannot affect the prices). In order for the economy to be in a stable situation the prices must be such that the total demand equals the total supply. This is the Walrasian competitive equilibrium. That it exists and is well defined in markets with a continuum of traders was shown by Aumann in 1966 [23]; moreover, unlike in finite markets, no convexity assumptions were required.

Another approach considers the possible trades that groups of agents – called coalitions – can make among themselves, in such a way that they all benefit. This leads to the core, a game theoretic concept that generalizes Edgeworth’s famous “contract curve”: the core consists of all those allocations that no coalition can improve upon. These are clearly different concepts:

The definition of competitive equilibrium assumes that the traders allow market pressures to determine prices and that they then trade in accordance with these prices, whereas that of core ignores the price mechanism and involves only direct trading between the participants. [16, p. 40]

Aumann showed in 1964 [16] that the core and the set of competitive allocations coincide in markets with a continuum of traders. By introducing the model of the continuum that expresses precisely the idea of perfect competition, he succeeded in making precise also this equivalence (originally suggested by Edgeworth<sup>3</sup> and proved in various other models),<sup>4</sup> which has since become one of the basic tenets of economic theory.

Aumann then turned to the study of other concepts in the context of perfectly competitive markets. A traditional idea in economics is that of “marginal worth” or “marginal contribution”. This idea is embodied in the concept of value, originally due to Shapley.<sup>5</sup> It may be interpreted as follows:

The Shapley value is an a priori measure of a game’s utility to its players; it measures what each player can expect to obtain, “on the average”, by playing the game. Other concepts of cooperative game theory ... predict outcomes (or sets of outcomes) that are in themselves stable, that cannot be successfully challenged or upset ... The Shapley value ... can be considered a mean, which takes into account the various power relationships and possible outcomes. [41, p. 995]

While the definition of competitive equilibrium or core generalizes in a straightforward manner to the continuum of players case, this is not so in the case of value. This led to a most prolific collaboration between Aumann and Lloyd Shapley, starting in the late sixties and culminating in 1974 with the publication of their book *Values of Non-Atomic Games* [I]. They addressed deep problems, both conceptual – how to define the correct notions – as well as technical, and solved them masterfully. As a consequence, most important and beautiful insights were obtained. One example is the “diagonal principle”, stating that in games with many players, one need consider only coalitions whose composition constitutes a good sample of the grand coalition of all participants. It is important to note that, unlike the core (or the competitive equilibrium), the value solution is applicable in almost every interactive setup. For instance, political contexts usually lead to situations where the core is empty, whereas the value is well-defined and yields most significant insights.

Returning to perfectly competitive economies, in 1975 [32] Aumann obtained another equivalence result, this time between the competitive allocations and the value allocations.<sup>6</sup> This is perhaps even more surprising than the core

<sup>3</sup>3. F. Y. Edgeworth, *Mathematical Psychics* (London: Kegan Paul, 1881).

<sup>4</sup>4. Most notably: M. Shubik, “Edgeworth Market Games”, in *Contributions to the Theory of Games*, vol. 4, ed. A. W. Tucker and R. D. Luce, Princeton University Press, 1959, pp.267-278; and G. Debreu and H. Scarf, “A Limit Theorem on the Core of an Economy,” *International Economic Review* 4 (1963), 236-246.

<sup>5</sup>5. L. S. Shapley, “A Value for n-Person Games”, in *Contributions to the Theory of Games*, vol. 2, ed. H. W. Kuhn and A. W. Tucker, Princeton University Press, 1953, pp. 343-359.

<sup>6</sup>6. Assuming the market is “sufficiently smooth”. Again, the continuum of traders model allows Aumann to obtain a precise and general result (the first such result is due to Shapley [L. S. Shapley, “Values of Large Games VII: A General Exchange Economy with Money”, RM-4248, the Rand Co., 1964, mimeo], in transferable utility markets only).

equivalence, since the concept of value does not capture, by its definition, considerations of stability and equilibrium.

This equivalence is indeed striking. In Aumann's view:

Perhaps the most remarkable single phenomenon in game and economic theory is the relationship between the price equilibria of a competitive market economy, and all but one of the major solution concepts for the corresponding game. ... Intuitively, the equivalence principle says that the institution of market prices arises naturally from the basic forces at work in a [perfectly competitive] market, (almost) no matter what we assume about the way in which these forces work. [55, p. 474]

This nicely exemplifies Aumann's view on the universality of the game theoretic approach:

to point out a fundamental difference between the game-theoretic and other approaches to social sciences. The more conventional approaches take institutions as given, and ask where they lead. The game theoretic approach asks how the institutions came about, what led to them? Thus general equilibrium theory takes the idea of market prices for granted; it concerns itself with their existence and properties, calculating them, and so on. Game Theory asks, *why* are there market prices? How did they come about? [55, p. 467]

The fundamental insights and understandings obtained in the analysis of perfect competition enabled and facilitated the study of basic economic issues that go beyond perfect competition. We mention a few where Aumann's contributions and influence are most noticeable: monopolistic and oligopolistic competition, modeled by a continuum of traders together with one or more large participants [28]; public economics – models of taxation based on the interweaving of the economic activities with a political process, such as voting [37, 38, 39, 44]; fixed-price models [51]; and others.

## 2. Repeated Games

Most relationships among rational decision makers last for a long time. Competition of firms in markets, insurance contracts, credit relationships, and negotiations, are often long term affairs. The same is true for employer-employee, client-lawyer, and firm-subcontractor relationships, as well as for conflicts and agreements between political parties and nations, or evolutionary processes in biology.

By their nature, the different stages of the game are interdependent in such long term interactions. This leads rational decision-makers to react to past experience, as well as to take into account the future impact of their choices. Many of the interesting and important patterns of behavior – like rewarding

and punishing, transmitting information and concealing it – can only be seen in multi-stage games. Foremost among these models are the repeated games, where the same game is played at each stage. This is a theory that Aumann has been instrumental in its systematical development.

Repeated games may be divided into two categories: repeated games of complete information, and repeated games of incomplete information. The two theories are, of course, strongly connected. Still, their focus may at times be different.

Repeated games of complete information assume that all the players know precisely the one-shot game that is repeatedly played.

The theory of repeated games of complete information is concerned with the evolution of fundamental patterns of interaction between people (or for that matter, animals; the problems it attacks are similar to those of social biology). Its aim is to account for phenomena such as cooperation, altruism, revenge, threats (self-destructive or otherwise), etc. – phenomena which may at first seem irrational – in terms of the usual “selfish” utility-maximizing paradigm of game theory and neoclassical economics. [42, p. 11]

The first result in this area emerged in the fifties, and its authorship is obscure; it is known as the “Folk Theorem”. It says that the strategic equilibrium payoffs of the repeated game coincide with the (jointly) feasible and individually rational payoffs of the one-shot game. This result may be viewed as relating noncooperative behavior in the multi-stage situation to cooperative behavior in the one-stage game. However,

while the set of all feasible individually rational outcomes does represent a solution notion of sorts for a cooperative game, it is relatively vague and uninformative. [42, p. 13]

Aumann therefore considers more specific cooperative behavior: the core. In 1959 [4] he defined the notion of a “strong equilibrium” – where no group of players can all gain by unilaterally changing their strategies – and showed that the strong equilibria of the repeated game correspond to the core (more precisely, the “beta-core”) of the one-shot game. It is noteworthy that this led Aumann to define and study “general” cooperative games – games with non-transferable utility – that turn out to be most important in economic theory (see the next section). Prior to this, only “side-payment” games – where each coalition can arbitrarily split among its members a fixed amount – were studied.

Another possible way to try and reduce the Folk Theorem set is suggested by Selten’s ideas of “perfectness”. Roughly speaking, perfectness implies that a player should not use “irrational threats” that, if carried out, may hurt him as well. A refinement of the Folk Theorem, due to Aumann and Shapley [62]<sup>7</sup> and

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<sup>7</sup>7. Originally written in 1976.

to Rubinstein,<sup>8</sup> in independent work, shows that this is not so: in the repeated game, the equilibrium outcomes and the perfect equilibrium outcomes coincide.

The other category of repeated games are those of incomplete information. Unlike the complete information case, here players may not possess some of the relevant information about the one-shot game that is repeatedly played; for example, a player may not know what the payoffs of the other players are. The importance of the repetition in this case is that it enables players to infer and learn some of the information of the other players from their behavior. More specifically, there is

a subtle interplay of concealing and revealing information: concealing, to prevent the other players from using the information to your disadvantage; revealing, to use the information yourself, and to permit the other players to use it to your advantage. [47, pp. 46-47]

The stress here is on the strategic use of information – when and how to reveal and when and how to conceal, when to believe revealed information and when not, etc. [42, p. 23]

Starting in the mid 1960s, Aumann, together with collaborators and also through his students, founded and developed the theory of repeated games with incomplete information. In the path-breaking reports<sup>9</sup> to the U.S. Arms Control and Disarmament Agency, Aumann and Michael Maschler set up in 1966 [V] the model of a repeated game with incomplete information. They showed that the complexity of the use of information alluded to above can actually be resolved in a beautiful, elegant, and explicit way. In the simplest case of a repeated two-person zero-sum game in which one player is more informed than the other (this is referred as “incomplete information on one-side”), they showed that the amount of information that is used (and revealed) by the informed player is precisely determined: at times, complete revelation or no revelation at all; and at other times, partial revelation. This analysis was then extended to more general models, zero-sum as well as non-zero-sum. It led to many new and deep ideas and concepts. For instance, Aumann, Maschler and Stearns introduced in 1968 [V] the notion of a “jointly controlled lottery”, a lottery where no player can unilaterally change the probabilities of the various outcomes; this turned out to be most relevant in the non-zero-sum setup.

The study of repeated games has greatly developed since this pioneering work. Aumann characterizes it as “a large, subtle and deep literature ..., which spilled over into related fields”. It has led to many important insights into the nature of incomplete information bargaining, and has been applied in many economic contexts, such as oligopoly, principal-agent, insurance, and many others.

At this point one should mention an additional interpretation of the repeated game model that Aumann propounds. He regards the repetition as a paradigm

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<sup>8</sup>A. Rubinstein, “Equilibrium in Supergames”, RM-26, Hebrew University of Jerusalem, 1976.

<sup>9</sup>These reports were written in 1966, 1967 and 1968. They are now collected in one book [V], together with extensive notes on the development of the theory since then.

for general bargaining procedures. The fact that the interaction is long term allows the participants to use single-shot actions – whose relative influence on the total payoff is negligible – for purposes like communication, signalling, and so on. Here again one sees Aumann’s “unified approach”. Rather than using different models that *exogenously* specify the bargaining and the communication among the players, a simple standard model is considered instead – that of the repeated game – where all these effects are obtained *endogenously*. Players are not forced by the rules of the game to negotiate. In a strategic equilibrium, they choose to do so.

### 3. Foundations

Game theory, as a discipline, is only half a century old (we just celebrated fifty years from the publication of *Games and Economic Behavior*). Its scope is universal, encompassing multi-participant interactive situations of all kinds. Its conceptual foundations are by no means clear. Indeed, they require deep understanding; what Aumann calls “comprehension”:

To come to grips with the question of what we are trying to do in game theory, we must first back off a little and ask ourselves what science in general is trying to do. ... On the most basic level, what we are trying to do in science is to understand our world. Predictions are an excellent means of testing our comprehension, and once we have the comprehension, applications are inevitable; but the basic aim of scientific activity remains the comprehension itself. ...

Comprehension is a complex concept, with several components. Perhaps the most important component has to do with fitting things together, relating them to each other ... reacting, associating, recognizing patterns. ... the second component of comprehension, which is really part of the first: unification. The broader the area that is covered by a theory, the greater is its ‘validity’. ... The third component of comprehension ... is simplicity. ... mostly the opposite of complexity, though the other meaning of “simple” – the opposite of “difficult” – also plays a role. [47, pp. 29-31]

Aumann has always devoted much of his effort to building the basic conceptual foundations of game theory. The very fact that there are many participants, whose decisions and reasoning are inter-dependent, may quickly make the analysis hopeless, if not lead to self-contradictions and paradoxes. Aumann succeeded in brilliantly resolving many of these problems. His pioneering contributions go deep into the heart of the issues involved, and always come out very clear, precise, and elegant.

The fundamental question, in particular in the interactive (multi-person) setup, is “what is rationality?”. Throughout the years Aumann developed and refined a unified view of this issue. At its basis lies the following statement:

“A player is rational if he maximizes his utility given his information.” Thus, a rational agent chooses an action that he prefers best; of course, “best” is taken relative to the knowledge he possesses (about the environment and the other participants). Surprisingly, this seemingly simple and clear statement may be understood in various different ways, some contradicting one another. It looks like the elusive Cheshire Cat that eats its tail – and vanishes with a grin. Indeed, what is “a player’s information”? What does he know about the others? About their rationality? Aumann has tackled these questions in a number of immensely influential works, that have set the standard for such models.

First, consider the issue of knowledge and information. Clearly, the behavior of each participant depends on what he knows. It then follows that it also depends on what he knows about what the others know about him. This cannot stop here; it is also relevant what each one knows about all this, that is, what each one knows about what each one knows about what each one knows. At this stage it all looks hopeless.

In 1976 [34], Aumann formalized the notion of *common knowledge*<sup>10</sup> that precisely captures this situation. He then showed that, for two agents who start with the same prior beliefs, if their posterior beliefs (based on different private information) about a specific event are common knowledge, then these posterior beliefs must necessarily agree. This paper had a huge impact. On the one hand, it led to the development of the whole area known today as “interactive epistemology”, that deals with the formal notions of knowledge in multi-person situations.<sup>11</sup> On the other hand, it found many applications, from economic models – such as the fact that there can be no trade between people with different information, so long as they have the same prior and their actions are commonly known<sup>12</sup> – to computer science – in the analysis of distributed environments, such as multi-processor networks, for instance.

Next, assume that the players are “Bayesian rational”, meaning that each one maximizes his utility with respect to his beliefs. This is standard in one-person decision theory.<sup>13</sup> What are however its implications in the multi-person setup? Aumann showed in 1987 [53] that it precisely corresponds to the notion of correlated equilibrium.

The underlying ... model ... starts by considering the set of all “states of the world,” where the specification of a state includes all relevant factors, including the (pure) strategy that each player uses in the given game, and what he knows when he decides to do so. It then asks, suppose that each player is rational at each state of the

<sup>10</sup>10. The philosopher D. K. Lewis was the first to clearly specify this concept: D. K. Lewis, *Convention: A philosophical Study*, Harvard University Press, 1969.

<sup>11</sup>11. See Aumann’s unpublished but widely circulated “Notes on Interactive Epistemology”, Yale University, 1989; also DP-67, Center for Rationality and Interactive Decision Theory, Hebrew University of Jerusalem, 1995.

<sup>12</sup>12. P. Milgrom and N. Stokey, “Information, Trade and Common Knowledge”, *Journal of Economic Theory* 26 (1982), 17-27.

<sup>13</sup>13. Aumann has contributed to this area as well; in particular, his pioneering paper with F. J. Anscombe [14] on subjective probability, and his work on utility theory [13].

world that occurs with positive probability; i.e., that the strategy that that state specifies for each player happens to be one that maximizes his utility given his information. ... The point of view of this model ... it asks, what are the *implications* of rationality in interactive situations? ... The answer given in Aumann 1987 was that it leads to correlated equilibrium – not that the players consciously choose a correlated equilibrium according to which they play, but that, to an outside observer with no information, the distribution of action n-tuples appears as if they had. [61, pp. 214-215]

The fundamental concept of *correlated equilibrium* was introduced by Aumann in 1974 [29]. It is a noncooperative equilibrium in a situation where players may use private information (“signals”) that they receive about some uncertain events. These signals may well be correlated; when they are stochastically independent, the classical Nash equilibrium obtains. Correlated equilibria appear in many contexts, economic and otherwise, and have led to further important studies on various communication procedures, and “mechanisms” in general.

In more recent work [65, 66], Aumann deals with the fundamental question of “What extent of rationality and knowledge of rationality is needed to obtain a strategic Nash equilibrium?” Contrary to what is commonly believed in the profession, the answer is *not necessarily* “common knowledge of rationality”.

Strict rationality is a demanding and complex assumption on the behavior of decision-makers. This led to considering models of *bounded rationality*, where this assumption is relaxed. In interactive situations, Aumann showed how “a little grain” of irrationality may go a long way. Indeed, it suffices in some cases to lead to cooperation in repeated games – as he obtained in his work with Sylvain Sorin [57]; and it resolves well-known “backward induction paradoxes” (such as the “centipede games”) [61]:

This attractive resolution for these paradoxes gives a rigorous justification to the elusive idea that, whereas one should certainly play rationally at the end, it seems somehow foolish to act from the very beginning in the most pathologically pessimistic, “play it safe” way. The above analysis shows that, contrary to what had been thought, one may act in a way that is rational – even with a high degree of mutual knowledge of rationality – and yet is quite profitable. And it reflects the fact that in real interactive situations there is a great deal of uncertainty about what the others will do, to what extent they are rational, what they think about what you think about your rationality, and so on. [61, pp. 222-223]

There are two traditional approaches to Game Theory: the *noncooperative* and the *cooperative*.

The non-cooperative theory concentrates on the strategic choices of the individual – how each player plays the game, what strategies he chooses to achieve his goals. The cooperative theory, on the other

hand, deals with the options available to the group – what coalitions form, how the available payoff is divided. It follows that the non-cooperative theory is intimately concerned with the details of the processes and rules defining a game; the cooperative theory usually abstracts away from such rules, and looks only at more general descriptions that specify only *what* each coalition can get, without saying *how*. A very rough analogy – not to be taken too literally – is the distinction between micro and macro, in economics as well as in biology and physics. Micro concerns minute details of process, whereas macro is concerned with how things look “on the whole”. Needless to say, there is a close relation between the two approaches; they complement and strengthen one another. ... noncooperative and cooperative game theory are two sides of the same coin [IV, preface of volume 1, p. xii; preface to volume 2, p. xv]

Up to this point we have discussed Aumann’s contributions to the foundations of noncooperative game theory. His work on the cooperative side is no less impressive.

Cooperative games were introduced by von Neumann and Morgenstern in their 1944 book. They defined the notion of a “game in coalitional form”, given by ascribing to each coalition of players a number. A standard interpretation – by no means the only one – views this number as the total payoff that the members of the coalition can arbitrarily divide among themselves. The underlying assumption is that there is a medium of exchange (“money”), which is freely transferable among the players, and such that each player’s utility is linear in it. These games are thus called games with “side-payments”, or with “transferable utility” (TU-*games*).

Aumann extended the classical TU theory to the general, *nontransferable* utility (NTU) case. He was led to it by his work on repeated games (see the previous section), and by the realization that this extension is useful in many applications where the TU assumption is not appropriate. He defined, first, the appropriate notion of an NTU coalitional form game, and second, the corresponding cooperative solution concepts (like the stable sets and the core) [5, 10, 24].

Throughout the years, Aumann has developed and strengthened the theory of cooperative games, both TU and NTU. On the one hand, he studied cooperative solution concepts in various models. We have already discussed the impressive results in purely competitive economic models, asserting the equivalence of cooperative solution concepts and the competitive equilibria. Among many other applications, we mention his work with Michael Maschler [46] on an interesting (and unexpected?) relation to a bankruptcy problem discussed in the Talmud,<sup>14</sup> where the nucleolus is obtained. The connection is through the basic postulate of consistency, “a remarkable property which, in one form or another, is common to just about all game-theoretic solution concepts” [55, p.

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<sup>14</sup>14. The Talmud forms the basis for Jewish civil, criminal, and religious law; it is about two thousands years old.

478]. Roughly speaking, it states that the solution of a sub-problem (defined in an appropriate manner from the original problem) should be the same as that of the original problem. All this is also connected to the concepts, introduced by Aumann and Maschler in 1964 [17], of proposals, objections and counter-objections, that led to new solution notions: the bargaining set, the kernel and the nucleolus.

On the other hand, he contributed to the foundations of cooperative game theory. One example is his axiomatization of the NTU-value. This solution concept, introduced by Shapley in 1967,<sup>15</sup> and extensively studied and applied by Aumann and others (see [48] for an extensive bibliography up to 1985), was defined constructively (by a fixed-point procedure). Unlike classical solutions like the Nash bargaining solution<sup>16</sup> and the Shapley (TU-)value,<sup>17</sup> it lacked an axiomatic foundation. Aumann succeeded in 1985 [45] in formulating a simple set of axioms that characterize the NTU-value. Once again, this paper opened the field and led to work that has ramified in many new directions.

Finally, one has to mention Aumann's philosophical papers and surveys [41, 42, 47, 55, 60]. They are no less influential than his more technical work. Indeed, Aumann succeeds in explaining even the most complex ideas and insights in a way that makes them accessible to a large and varied audience. His surveys are not just a dry list of results. Rather, they exhibit the grand picture in all its beauty and clarity. They point out the accomplishments, and also the difficulties that need to be addressed and the areas of future research. There is no doubt that Aumann's viewpoints in general, and these papers in particular, have played an essential role in establishing and sharpening the game and economic theoretic thinking.

#### 4. Other Contributions

We have collected here a number of important additional contributions of Aumann. The first concerns set-valued functions (or "correspondences"), namely functions whose values are sets of points rather than a single point. Aumann made many important contributions [8, 11, 20, 21, 26] in this area, such as the "Aumann measurable selection theorem", results on integrals of set-valued functions, and so on. Most of these problems were motivated by the study of various game theoretic and economic models, in particular those with a continuum of agents, and the mathematical theory developed was instrumental in the evolution and analysis of these models. The results obtained by Aumann are fundamental, and they are used in many areas in economics, mathematics and operations research, such as general equilibrium, optimal allocation, non-linear

<sup>15</sup>15. L. S. Shapley, "Utility Comparison and the Theory of Games," in *La Decision* (Paris: Editions du C.N.R.S., 1969), pp. 251-263.

<sup>16</sup>16. J. F. Nash, "The Bargaining Problem," *Econometrica* 18 (1950): 155-162.

<sup>17</sup>17. L. S. Shapley, "A Value for n-Person Games," in *Contributions to the Theory of Games*, vol. 2, ed. H. W. Kuhn and A. W. Tucker (Princeton: Princeton University Press, 1953), pp. 343-359.

programming, control theory, measure theory, fixed-point theorems, and so on.

Kuhn's well known result on the equivalence of behavioral and mixed strategies in finite games of perfect recall was extended by Aumann [18] to the infinite case, while overcoming complex technical difficulties.

In cooperative games, players may organize themselves into coalitions. Aumann has studied such games (with Jacques Drèze [31]), and also models leading to the formation of coalitions (with Roger Myerson [56]).

This concludes our survey – by no means complete – of Aumann's main contributions. One cannot help at this point but be greatly impressed by the “grand picture” that emerges.

### **Aumann's Students**

We have seen up to now some of Aumann's path-breaking contributions. But this is only part of the picture. In addition to his published work, Aumann had throughout the years a considerable direct impact on the research of many people. He suggested to them important problems and avenues of research, shared with them his deep insights and understandings, and helped and encouraged them throughout their work. Foremost among these are, of course, his students.

Aumann always directed his students to central and difficult problems in the field. Their solution most often led to new important insights. A key feature of the interaction between Aumann and his students was the two-way feedback: the topic of research was usually an essential building block in Aumann's world, and the results obtained were then used by him to shape and refine his views and understandings.

Aumann applied his very high scientific standards to his students' work as well. He always required one to obtain complete and precise results, that identify and establish the correct relations between different notions, and delineate the exact framework within which these hold.

Aumann has had, up to date (1995), twelve doctoral students (see the list in Appendix C); almost all are now leading researchers in their own right. In addition, Aumann supervised many Master's theses, some of which resulted in published papers as well.

Aumann has always been most proud of his students. And, of course, his students have always been very grateful for the opportunity of working with Aumann and sharing in the wonderful world of scientific research and discovery.

## This Volume

Aumann is now (June 1995) sixty-five years old. We have looked for an appropriate way to commemorate this event, in a manner that will shed light on his great impact on game theory and economic theory. We have therefore selected some of the outstanding published papers of his doctoral students and collected them in this volume.

These are important and mostly well-known contributions, covering many of the areas of game and economic theory, in particular some that are very close to Aumann's world. The main criterion of selection was the prominence and significance of the results. It is thus a "Best of" collection; the papers chosen have proved their importance throughout the years. A few of these were directly influenced by Aumann (for example, they came out of the doctoral theses). Others were indirectly influenced by his thinking and ideas, and some may seem only remotely related to Aumann's work. But there is no doubt that the scientific "education" received from Aumann has been instrumental in all of them.

It is our hope that this collection of selected important contributions will further illuminate the far-reaching impact of Robert J. Aumann.

The volume consists of twenty-two papers, two by each one of his doctoral students.<sup>18</sup> We have organized them by topics in six parts.<sup>19</sup>

We now briefly survey the main contribution of each paper, and try to show its place and importance in the area, as well as its relation to the work of Aumann that we have discussed above.

The first part I gathers three basic contributions to the concept of *strategic equilibrium*. Originally due to Nash, the noncooperative equilibrium was "refined" by Selten, who introduced the notions of "perfectness". Kalai and Samet (Chapter 1) consider sets of strategies that satisfy the same local stability property that is required of single strategies in a perfect equilibrium. This leads to the concept of *persistent equilibria*, which is defined and studied in this paper. In particular, it is shown that every strategic game has an equilibrium that is perfect, proper and persistent. Kohlberg and Mertens (Chapter 2) introduce the first notion of a *strategically stable set of equilibria* of a game: a set of equilibria such that all nearby games have equilibria close to that set. This contribution refocused the literature on strategic equilibrium and its refinements, and it goes at least part of the way toward an axiomatic approach; that is, formulating postulates that stable equilibria should satisfy. The paper also obtains important results on the structure of the equilibrium correspondence (namely, that it is a "deformation" of a constant map). Kalai and Lehrer (Chapter 3) examine the question of the process by which players select the strategies they play. A natural *learning* model is introduced, where the players update their beliefs about their opponents. It is shown that this leads, in the long-run, to a play that is

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<sup>18</sup>18. Eleven of the twelve doctoral students are active researchers in the area.

<sup>19</sup>19. Part II could be further subdivided into two subparts, "repeated games" and "stochastic games." The chapters are ordered by date in each (sub)part.

close to a strategic equilibrium. It is worthwhile to mention that the tools and ideas stem from the theory of repeated games with incomplete information.

Part II contains contributions to “dynamic” noncooperative games. These are long-run multi-stage games, with a certain kind of stationary time structure. The most basic ones are the *repeated games* – with and without *complete information* – and the *stochastic games*. Aumann and Maschler [V] pioneered the study of repeated games of incomplete information, and showed that the value of a two-person zero-sum repeated game with incomplete information on one side exists and is well-defined. Moreover, it is the same whether one looks at the limit of the values of the finitely repeated games (as the number of repetitions increases), or at the value of the limit, infinitely repeated game. However, this is no longer so in the *two-sided information* case, where each one of the two players lacks some information. Aumann, Maschler and Stearns [V] showed that the infinitely repeated game may indeed have no value in this case. The open problem of whether the values of the finitely repeated games do converge to some limit was solved by Mertens and Zamir (Chapter 4). They introduce a functional equation, prove that it has a solution, and show that this is precisely the limit of the values of the finitely repeated games. Next, consider two-person *non-zero-sum* repeated games of incomplete information. The simplest case is, again, when the information is one-sided. Here Aumann, Maschler and Stearns [V] showed that the equilibria may be much more complex than in the zero-sum case. Hart (Chapter 5) provides a complete characterization of all the equilibria of such games. In the zero-sum case, there is always just one step of “signaling”, that is, revelation of information by the informed player. In contrast, in the non-zero-sum case it is shown that “many” stages of communication between the players are needed; in particular, information is at times revealed little-by-little. The third contribution in this part deals with repeated games of complete information with *imperfect monitoring*, where – unlike the standard models – the actions of a player are not directly observable by his opponent. Lehrer (Chapter 6) studies the correlated equilibria in two-player repeated games with nonobservable actions, and obtains complete characterizations of the corresponding equilibrium sets. These depend on the way the players evaluate their payoffs in the game, and are based on the notion of the relative degree of “informativeness” of an action: how much it reveals about the actions of the opponent.

The last two contributions in part B are to the theory of *stochastic games*. Stochastic games were introduced in 1953 by Shapley,<sup>20</sup> who completely solved the two-person zero-sum discounted case. In the mid seventies, interest and research activity in stochastic games was further motivated by the study of repeated games of incomplete information; for instance, one may view the uncertainty as a state variable. Bewley and Kohlberg (Chapter 7) study the *asymptotic behavior* of the minimax value of two-person zero-sum stochastic games, both as the discount rate becomes small, and as the number of stages becomes large. They show that the two limits exist and are identical. Moreover, they

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<sup>20</sup>20. L. S. Shapley, “Stochastic Games”, *Proceedings of the National Academy of Sciences of the U.S.A.* 39 (1953), 1095-1100.

prove that the value – as well as the optimal strategies – of the discounted game has a convergent expansion in fractional powers of the discount rate. Mertens and Neyman (Chapter 8) solve the problem of the existence of the minimax value for *undiscounted* two-person zero-sum stochastic games. They prove that all such games have a value, and moreover in a strong sense: each player has a strategy that is almost optimal in all sufficiently long finite games (as well as in the infinite game). Furthermore, their result goes beyond the finite case, where the state space and the actions sets are all finite.

Part III consists of contributions to cooperative game theory. Schmeidler (Chapter 9) introduces a new cooperative solution concept – the *nucleolus*. It is a direct descendent of the “bargaining set” and the “kernel”, stemming from the ideas of a “justified objection” of Aumann and Maschler [17], and recently appearing unexpectedly in their study [46] of a bankruptcy problem from the Talmud. The paper defines the nucleolus, and proves that it always consists of a unique outcome; moreover, it belongs to the kernel and to the core, if the latter is non-empty. The next two chapters deal with fundamental questions raised by Aumann and Shapley [I] in their study of *values of nonatomic games*. One most important open problem was whether the *diagonal property* is a consequence of the axioms of value. Indeed, all values turned out to satisfy this property, namely, that the value of a game depends only on the worth of those coalitions that are an almost-perfect sample of the whole population. This seems a reasonable condition in view of the “large numbers” aspect of nonatomic games. The question was whether it is indeed always satisfied. Tauman (Chapter 10) provides a counter-example to this conjecture in a natural “reproducing space”, a space generated by monotonic games. Another most difficult question that was left open was whether “jump-function” games, such as majority voting games, have an *asymptotic value*. The significance of the asymptotic approach to value is that it captures the idea that the nonatomic game is the limit of large but finite games. Neyman (Chapter 11) succeeded in solving this problem. He stated and proved a renewal theorem for sampling without replacement. The theorem is equivalent to the existence of an asymptotic value for the simplest nonatomic weighted majority games, and it further enabled the generalization of this result to a large and important class of games. The last contribution in this part, by Peleg (Chapter 12), provides an axiomatization for the *core* of general nontransferable utility cooperative games. The core is a most basic cooperative solution concept, which has moreover been extensively applied, in particular in economics. The axiomatization uses the property of *consistency* (or, the *reduced game property*) that is satisfied, in one form or another, by most of the solution concepts in game theory. Such axiomatizations provide ways to better grasp the common and the different among various solutions and thus understand what each one “means”.

The contributions in Part IV all relate to economic models. The first three chapters address the extent to which Aumann’s [16, 32] *equivalence* theorems for *purely competitive* economies hold. Starting with the *core*, Shitovitz (Chapter 13) shows that perfect competition may arise even when there are relatively large participants, modeled as atoms in a nonatomic continuum of negligible

agents. He identifies conditions under which any core allocation is competitive. Two such cases are, first, when there is more than one large trader and they all have the same characteristics; and second, when the large traders can be divided into two or more identical (or similar) groups. In all these cases it turns out that the competition between the large traders is strong enough to guarantee that they can make no gains relative to the competitive outcome. The next paper of Shitovitz (Chapter 14) provides a further simple demonstration of this seemingly surprising result; moreover, it turns out to apply to more general models. The other equivalence result in markets with a continuum of traders obtained by Aumann [32] is between the *value* and the competitive allocations. For this result – unlike the core equivalence – smoothness assumptions were needed. Hart (Chapter 15) investigates the general case where no differentiability is assumed. The asymptotic approach is used, since the value is now no longer uniquely defined. It is proved that only one part of the value equivalence result holds; namely, that value allocations are always competitive, but that the converse is no longer true in general. The last contribution in this part, by Mirman and Tauman (Chapter 16), deals with the problem of cost allocation; that is, determining the share of each activity in the total cost of a complex organization. This seems at first to be unrelated to large economies: the number of activities may well be small. However, once the cost allocation problem is set up axiomatically – that is, natural conditions on the sharing rule are imposed – it yields, as shown in this paper, precisely the value of an appropriate nonatomic game. To understand the connection, think of each infinitesimal part of each activity as a “player”. The result may be viewed as an extension of the classical “average cost pricing” to the multi-product nonseparable case.

Part V consists of three selected contributions to the fundamental ingredients of the theory of interactive decisions: information, knowledge, and utility. To deal with the problem of modeling incomplete information, Harsanyi<sup>21</sup> postulated that every player is of one of several *types*, each type corresponding to a possible preference ordering for that player, together with a subjective belief (i.e., a probability distribution) over the types of the other players. From this one may indeed deduce the whole “hierarchy of beliefs” of the players; that is, what each one believes about the preferences of the others, and also about their beliefs about his own beliefs, and so on. Mertens and Zamir (Chapter 17) formalize Harsanyi’s fundamental idea and show that his model of games of incomplete information is indeed universal. Specifically, they prove that, given a hierarchy of beliefs, a “universal belief set” can always be constructed, to serve as the set of types for each player. Next, consider the basic concept of *common knowledge*, which is essential to much of the foundations of interactive decision theory. However, there are situations where it is impossible to achieve common knowledge, and furthermore any finite order mutual knowledge<sup>22</sup> turns out to be significantly distinct from the full common knowledge (the classical

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<sup>21</sup>21. J. C. Harsanyi, “Games of Incomplete Information Played by Bayesian Players, Parts I, II and III,” *Management Science* 14 (1967-68), 159-182, 320-334, 486-502.

<sup>22</sup>22. That is, all statements of the form “A knows that B knows that ... Z knows” with a fixed finite number of “knows,” hold.

example – from the computer science literature – has acknowledgement messages that go back and forth, but also may get lost). This raises the question of what the appropriate approximation to common knowledge is, if such exists at all. Monderer and Samet (Chapter 18) introduce the notion of *common belief* – “knowledge” is replaced by “belief with high probability” – and show that it is a right approximation. The behavior of players induced by common beliefs and by common knowledge are indeed close. The third contribution in this part, by Schmeidler (Chapter 19), concerns the foundation of one-person decision theory. It extends the standard Anscombe-Aumann [14] model, to allow for a von Neumann-Morgenstern expected utility with respect to a *nonadditive subjective probability*. This is obtained axiomatically by weakening the “monotonicity axiom,” and it is consistent with situations that contradict the classical additive expected utility theory.

Part VI starts with a contribution by Peleg (Chapter 20) to *social choice*. The fundamental result of Gibbard and Satterthwaite<sup>23</sup> states that in every voting scheme, either there is a “dictator”, or there are instances where participants are motivated to misrepresent their preferences. However, if this misrepresentation does not lead to a different social outcome, then the voting scheme is still of interest; it is called “consistent”. In this paper one deals with manipulations by coalitions; thus, the Nash strategic equilibrium is replaced by the Aumann [4] strong equilibrium. It is shown that such “strongly consistent” voting schemes do exist; moreover, a procedure for constructing them is provided, as well as a necessary and sufficient condition for their existence. The last two contributions deal with the theory of *set-valued functions*, or *correspondences*. Aumann [21] defined the integral of a correspondence as the set of the integrals of all its integrable selections. In many applications – for instance, in control theory and in mathematical economics – the correspondence depends measurably on a parameter. Artstein (Chapter 21) shows that in this case a measurable selection may be chosen to be jointly measurable in the parameter and the independent variable. Among other applications, this leads to an asymptotic result for decisions in large teams. An important property of the Aumann integral over a nonatomic measure is that it yields a convex compact set. Artstein (Chapter 22) interprets this as a “bang-bang” theorem, which says that the whole range is generated by the extreme points. He also provides a discrete analog of this result. All this implies and connects a large number of important results – known as well as new – in a variety of fields. Among these are the Shapley-Folkman lemma on the near convexity of large sums of sets, the theorem of Lyapunov on the convexity of the range of a nonatomic vector measure, the Dvoretzky-Wald-Wolfowitz theorem, Caratheodory’s theorem, a moment problem in statistics, results in linear control systems and a minimum norm problem in optimization.

This concludes the survey of the contents of this volume.

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<sup>23</sup>23. A. Gibbard, “Manipulation of Voting Schemes: A General Result”, *Econometrica* 47 (1973), 75-80; M. A. Satterthwaite, “Strategy-Proofness and Arrow’s Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions,” *Journal of Economic Theory* 10 (1975), 187-217.

## Appendix A: A Short Biographical Sketch of Robert J. Aumann

Robert J. Aumann (also known as Bob, Johnny and Yisrael)<sup>24</sup> was born in Frankfurt-am-Main in Germany on June 8, 1930. The Aumann family left Germany in 1938 and moved to New York. Bob Aumann got his B.Sc. in Mathematics in 1950 at the City College of New York, and then went on to graduate studies at the Massachusetts Institute of Technology (M.I.T.), where he got his Ph.D. in Mathematics in 1955. His thesis was in algebraic topology; more precisely, the theory of “knots”.

After finishing his doctorate, Aumann joined the Princeton University group that worked on industrial and military applications. Here he realized the importance and relevance of Game Theory, which was then in its infancy.

In 1956 Aumann immigrated to Israel<sup>25</sup> and was appointed Instructor in the Institute of Mathematics of the Hebrew University of Jerusalem. Promoted to Professor in 1968, he is to this day a member of this department. Throughout the years he visited many institutions. Longer visits include Princeton University, Yale University, University of California at Berkeley, Université Catholique de Louvain (Belgium), Stanford University, University of Minnesota, Mathematical Sciences Research Institute at Berkeley, State University of New York at Stony Brook, U.S. National Bureau of Standards and the Rand Corporation. He has also been associated for almost twenty-five years, as an “outside teacher”, with the Departments of Statistics and Mathematics of Tel-Aviv University.

Robert J. Aumann has been a Member of the U.S. National Academy of Sciences since 1985, a Member of the Israel Academy of Sciences and Humanities since 1989, and a Foreign Honorary Member of the American Academy of Arts and Sciences since 1974. He was the recipient of the Israel Prize in Economics in 1994, and of the Harvey Prize in Science and Technology (awarded by the Israel Institute of Technology) in 1983. He was awarded Honorary Doctorates by the University of Bonn in 1988, by the Université Catholique de Louvain in 1989, and by the University of Chicago in 1992. He was elected Fellow of the Econometric Society in 1966, and served for a number of years on its Council and its Executive Committee. Aumann was the President of the Israel Mathematical Union, and is an Honorary Member of the American Economic Association. Recipient, 1998 Erwin Plein Nemmers Prize in Economics.

His editorial appointments include *International Journal of Game Theory*, *Journal of Mathematical Economics*, *Journal of Economic Theory*, *Econometrica*, *Mathematics of Operations Research*, *SIAM Journal on Applied Mathematics* and *Games and Economic Behavior*.

Aumann was the organizer, together with Michael Maschler, of the First International Workshop in Game Theory in Jerusalem in 1965, and he organized

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<sup>24</sup>24. His wife Esther tells the story of the difficulties she once had while issuing passports for their children; the clerk could not understand why each one had a different father!

<sup>25</sup>25. Actually, he realized that his preferred choice was to go to Jerusalem while on the phone with another institution telling them that he was accepting their offer. This may explain why he always stresses the distinction between what one “does” and what one “says he would have done” (as in empirics versus experiments).

the Emphasis Year on Game Theory and Mathematical Economics at the Institute of Advanced Studies of the Hebrew University in Jerusalem in 1979-1980.

Robert J. Aumann delivered major invited addresses at many international congresses, including the Second World Congress of the Econometric Society (Cambridge, 1970), the Winter Meeting of the Econometric Society (the Walras-Bowley Lecture, Toronto, 1972), the International Congress of Mathematicians (Helsinki, 1978), the Fourth World Congress of the Econometric Society (Aix-en-Provence, 1980), the Ninth International Congress on Logic, Philosophy, and Methodology of the Sciences (Uppsala, 1991), and the Southern European Economic Association (Alfredo Pareto Lecture, Toulouse, 1992).

Aumann has been most successful throughout the years in attracting many good students and scholars to the field. Twelve Doctoral (Ph.D.) students did their research under his supervision; they are listed in Appendix C. In addition, he has had very successful Master (M.Sc.) students, whose theses have often resulted in published papers. Finally, there are many others whose work he has influenced in a most direct way, even though they were not officially his students.

## Appendix B: Publications of Robert J. Aumann

### Books

- [I ] Values of Non-Atomic Games, Princeton University Press, 1974, xi + 333 pp. (with Lloyd S. Shapley).
- [II ] Game Theory (in Hebrew), Everyman's University, Tel-Aviv, 1981, Vol 1: 211 pp., Vol 2: 203 pp. (with Yair Tauman and Shmuel Zamir).
- [III ] Lectures on Game Theory, Underground Classics in Economics, Westview Press, Boulder, 1989, ix + 120 pp.
- [IV ] Handbook of Game Theory with Economic Applications, Elsevier / North-Holland, Amsterdam, Vol 1, 1982: xxvi + 733 pp.; Vol. 2, 1994: xxviii + 786 pp.; Vol. 3, forthcoming (coedited with Sergiu Hart).
- [V ] Repeated Games of Incomplete Information, M.I.T. Press, 1995 (with Michael Maschler and the collaboration of Richard E. Stearns).

### Articles

- [1] "Asphericity of Alternating Knots", *Annals of Mathematics* 64 (1956):374-392.
- [2] "The Coefficients in an Allocation Problem", *Naval Research Logistics Quarterly* 5(1958), pp. 111-123 (with J. B. Kruskal).

- [3] "Assigning Quantitative Values to Qualitative Factors in the Naval Electronics Problem", *Naval Research Logistics Quarterly* 6 (1959), pp. 1-16 (with J. B. Kruskal).
- [4] "Acceptable Points in General Cooperative n-Person Games", in *Contributions to the Theory of Games IV*, Annals of Math. Study 40, Princeton University Press, 1959, pp. 287-324.
- [5] "Von Neumann - Morgenstern Solutions to Cooperative Games Without Side Payments", *Bulletin of the American Mathematical Society* 66 (1960), pp. 173-179 (with Bezalel Peleg).
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- [7] "A Characterization of Game Structures of Perfect Information", *Bulletin of the Research Council of Israel* 9F (1960), pp. 43-44.
- [8] "Spaces of Measurable Transformations", *Bulletin of the American Mathematical Society* 66 (1960), pp. 301-304.
- [9] "Linearity of Unrestrictedly Transferable Utilities", *Naval Research Logistics Quarterly* 7 (1960), pp. 281-284.
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- [11] "Borel Structures for Function Spaces", *Illinois Journal of Mathematics* 5 (1961), pp. 614-630.
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- [13] "Utility Theory Without the Completeness Axiom", *Econometrica* 30 (1962), pp. 445-462.
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- [14] "A Definition of Subjective Probability", *Annals of Mathematical Statistics* 34 (1963), pp. 199-205 (with Frank J. Anscombe).
- [15] "On Choosing a Function at Random", in *Ergodic Theory*, Academic Press, 1963, pp. 1-20.
- [16] "Markets with a Continuum of Traders", *Econometrica* 32 (1964) pp. 39-50.
- [17] "The Bargaining Set for Cooperative Games", in *Advances in Game Theory*, Annals of Mathematics Study 52, Princeton University Press, 1964, pp. 443-476 (with Michael Maschler).
- [18] "Mixed and Behavior Strategies in Infinite Extensive Games", in *Advances in Game Theory*, Annals of Mathematics Study 52, Princeton University Press, 1964, pp. 627-650.
- [19] "Subjective Programming", in *Human Judgments and Optimality*, John Wiley and Sons, New York, 1964, pp. 217-242.

- [20] “A Variational Problem Arising in Economics”, *Journal of Mathematical Analysis and Applications* 11 (1965), pp. 488-503 (with Micha Perles).
- [21] “Integrals of Set-Valued Functions”, *Journal of Mathematical Analysis and Applications* 12 (1965), pp. 1-12.
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- [23] “Existence of Competitive Equilibria in Markets with a Continuum of Traders”, *Econometrica* 34 (1966), pp. 1-17.
- [24] “A Survey of Cooperative Games Without Side Payments”, in *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, Princeton University Press, 1967, pp. 3-27.
- [25] “Random Measure Preserving Transformations”, in *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Volume II, Part 2, University of California Press, 1967, pp. 321-326.
- [26] “Measurable Utility and the Measurable Choice Theorem”, in *La Décision*, pp. 15-26. Paris: Editions du Centre National de la Recherche Scientifique, 1969,
- [27] “Some Thoughts on the Minimax Principle”, *Management Science* 18 (1972), pp. P-54–P-63 (with Michael Maschler).
- [28] “Disadvantageous Monopolies”, *Journal of Economic Theory* 6 (1973), pp. 1-11.
- [29] “Subjectivity and Correlation in Randomized Strategies”, *Journal of Mathematical Economics* 1 (1974), pp. 67-96.
- [30] “A Note on Gale’s Example”, *Journal of Mathematical Economics* 1 (1974), pp. 209-211 (with Bezalel Peleg).
- [31] “Cooperative Games with Coalition Structures”, *International Journal of Game Theory* 4 (1975), pp. 217-237 (with Jacques H. Drèze).
- [32] “Values of Markets with a Continuum of Traders”, *Econometrica* 43 (1975), pp. 611-646.
- [33] “An Elementary Proof that Integration Preserves Uppersemicontinuity”, *Journal of Mathematical Economics* 3 (1976), pp. 15-18.
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- [36] “The St. Petersburg Paradox: A Discussion of some Recent Comments”, *Journal of Economic Theory* 14 (1977), pp. 443-445.
- [37] “Power and Taxes”, *Econometrica* 45(1977), pp. 1137-1161 (with Mordecai Kurz).
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- [39] “Core and Value for a Public Goods Economy: An Example”, *Journal of Economic Theory* 15 (1977), pp. 363-365 (with Roy J. Gardner and Robert W. Rosenthal).
- [40] “On the Rate of Convergence of the Core”, *International Economic Review* 19 (1979), pp. 349-357.
- [41] “Recent Developments in the Theory of the Shapley Value”, in *Proceedings of the International Congress of Mathematicians*, Helsinki, 1978, Academia Scientiarum Fennica, 1980, pp. 995-1003.
- [42] “Survey of Repeated Games”, in *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern*, Vol. 4 of Gesellschaft, Recht, Wirtschaft, Wissenschaftsverlag, Bibliographisches Institut, Mannheim, 1981, pp. 11-42.
- [43] “Approximate Purification of Mixed Strategies”, *Mathematics of Operations Research* 8 (1983), pp. 327-341 (with Yitzhak Katznelson, Roy Radner, Robert W. Rosenthal and Benjamin Weiss).
- [44] “Voting for Public Goods”, *Review of Economic Studies* 50 (1983), pp. 677-694 (with Mordecai Kurz and Abraham Neyman).
- [45] “An Axiomatization of the Non-Transferable Utility Value”, *Econometrica* 53 (1985), pp. 599-612.
- [46] “Game Theoretic Analysis of a Bankruptcy Problem from the Talmud”, *Journal of Economic Theory* 36 (1985), pp. 195-213 (with Michael Maschler).
- [47] “What Is Game Theory Trying to Accomplish?”, in *Frontiers of Economics*, edited by K. J. Arrow and S. Honkapohja, Basil Blackwell, Oxford, 1985, pp. 28-76.
- [48] “On the Non-Transferable Utility Value: A Comment on the Roth-Shafer Examples”, *Econometrica* 53 (1985), pp. 667-677.
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[63] “The Shapley Value”, in *Game-Theoretic Methods in General Equilibrium Analysis*, edited by J.-F. Mertens and S. Sorin, Kluwer Academic Publishers, 1994, pp. 61-66.

[64] “Economic Applications of the Shapley Value”, in *Game-Theoretic Methods in General Equilibrium Analysis*, edited by J.-F. Mertens and S. Sorin, Kluwer Academic Publishers, 1994, pp. 121-133.

[65] “Backward Induction and Common Knowledge of Rationality”, *Games and Economic Behavior* 8 (1995), pp. 6-19.

[66] “Epistemic Conditions for Nash Equilibrium”, *Econometrica*, forthcoming (with Adam Brandenburger).

#### **Reprinted Publications (including revisions, translations, etc.)**

[A] *Values of Non-Atomic Games* (Russian translation of Book I, Mir, Moscow, 1977).

[B] *Lectures on Game Theory* (Japanese translation of Book III), Keso-Shobo, Tokyo, 1991.

[C] “Almost Strictly Competitive Games” (Paper 12), in *Game Theory and Related Approaches to Social Behavior*, edited by M. Shubik, John Wiley and Sons, New York, 1964.

- [D] “Markets with a Continuum of Traders” (Paper 16), in *Readings in Mathematical Economics*, edited by P. Newman, The Johns Hopkins Press, Baltimore, 1968; also in *Mathematical Economics*, Mir, Moscow, 1974 (Russian Translation); also in *Game Theory in Economics*, edited by A. Rubinstein, Edward Elgar, Aldershot and Brookfield, 1990.
- [E] “Existence of Competitive Equilibria in Markets with a Continuum of Traders,” (Paper 23), in *Mathematical Economics*, Mir, Moscow, 1974 (Russian translation).
- [F] “Agreeing to Disagree”, (Paper 34), in *Game Theory in Economics*, edited by A. Rubinstein, Edward Elgar, Aldershot and Brookfield, 1990.
- [G] “Power and Taxes in a Multi-Commodity Economy (Updated)” (Paper 38, updated version), *Journal of Public Economics* 9 (1978), pp. 139-161.
- [H] “Repeated Games” (Paper 42, updated version), in *Issues in Contemporary Microeconomics and Welfare*, edited by G. R. Feiwel, Macmillan, London, 1985.
- [I] “The Shapley Value”, (an adaptation of a part of Paper 47), in *Game Theory and Applications*, edited by T. Ichiishi, A. Neyman and Y. Tauman, Academic Press, San Diego, 1990, pp. 158-165.
- [J] “Values of Markets with Satiation or Fixed Prices”, (Paper 51, abridged version), in *Underemployment Equilibria*, by J. H. Drèze, Cambridge University Press, Cambridge, 1991, pp. 111-131.
- [K] “Correlated Equilibrium as an Expression of Bayesian Rationality” (Paper 53), in *Game Theory in Economics*, edited by A. Rubinstein, Edward Elgar, Aldershot and Brookfield, 1990.
- [L] “Game Theory” (Paper 55), in *The New Palgrave Game Theory*, edited by J. Eatwell, M. Milgate and P. Newman, Macmillan, London & Basingstoke, 1989, pp. 1-53.

### Other Publications

- [a] “The Game of Politics”, *World Politics* 14 (1962), pp. 675-686. (A review of Anatol Rapaport’s “Fights, Games, and Debates”.)
- [b] Introduction to “Some Thoughts on the Theory of Cooperative Games”, by Gerd Jentzsch, in *Advances in Game Theory*, Annals of Mathematics Study 52, Princeton University Press, 1964, pp. 407-409.
- [c] “Game Theory: A Game?” (in Hebrew), an interview, *Makhshavot* (Thoughts), a publication of IBM Israel, Number 29 (February 1970), p. 9ff.
- [d] “Economic Theory and Mathematical Method: An Interview”, in *Arrow and the Ascent of Modern Economic Theory*, edited by G. R. Feiwel, Macmillan, London, 1987, pp. 306-316.

- [e] “Arrow – the Breadth, Depth, and Conscience of the Scholar: An Interview”, in *Arrow and the Foundations of the Theory of Economic Policy*, edited by G. R. Feiwel, Macmillan, London, 1987, pp. 658-662.
- [f] “Letter to Leonard Savage, 8 January 1971”, in *Essays on Economic Decisions under Uncertainty*, by J. H. Drèze, Cambridge University Press, Cambridge, 1987, pp. 76-78.
- [g] Report of the Committee on Election Procedures for Fellows, *Econometrica* 55, 1987, pp. 983-988 (with Michael Bruno, Frank Hahn and Amartya Sen).
- [h] Foreword to *A General Theory of Equilibrium Selection in Games*, by John C. Harsanyi and Reinhard Selten, The MIT Press, Cambridge and London, 1988, pp. xi-xiii.
- [i] Foreword to *Two-Sided Matching, a Study in Game-Theoretic Modeling and Analysis*, by Alvin E. Roth and Marilda A. Sotomayor, Cambridge University Press, Cambridge, 1990, p. xi.

Appendix C: The Doctoral Students of Robert J. Aumann  
(In chronological order)

Bezalel Peleg  
 David Schmeidler  
 Shmuel Zamir  
 Benyamin Shitovitz  
 Zvi Artstein  
 Elon Kohlberg  
 Sergiu Hart  
 Eugene Wesley  
 Abraham Neyman  
 Yair Tauman  
 Dov Samet  
 Ehud Lehrer

Game theory concepts have become hugely important in economics and whilst there are limits to how far analysis can stretch, making use of some of the core ideas can add greatly to your evaluation and analysis marks in the summer papers. Theory of the firm. Day to day pricing decisions taken by firms. Falling oil prices – if one airline decides to lower their fares as oil prices decline, will others follow? – Game Theory (A Level Economics Revision Update 2018) - revision video. Print page. Share – Adam Smith, Karl Marx and Friedrich Hayek on Economic Systems. Study notes. Trade Liberalisation. Game theory is the study of mathematical models of strategic interaction among rational decision-makers. It has applications in all fields of social science, as well as in logic, systems science and computer science. Originally, it addressed zero-sum games, in which each participant's gains or losses are exactly balanced by those of the other participants. In the 21st century, game theory applies to a wide range of behavioral relations, and is now an umbrella term for the science of logical decision Game theory attempts to take into consideration the interactions between the participants and their behavior to study the strategic decision-making between rational individuals. It tries to find out the actions that a –player– should perform which would maximize his chances of success mathematically and logically. Actions by everyone involved directly alter dynamics of the game, and hence the players are all interdependent. – By mathematically proving that an equilibrium point exists, John Nash showed that important economic, political or social interactions can be hinged on desirable outcomes without the need for any contracts. Examples of Game Theory. There are multiple real-life examples for understanding the basic concept of game theory. We examine the state of game theory. Many important questions have been answered, and game theoretic methods are now central to much economic investigation. We suggest areas where further advances are important, and argue that models of learning and of social preferences provide promising routes for improving and widening game theory’s predictive power, while preserving the successes of existing theory where it works well. We emphasize in particular the need for better understanding of the speed with which learning takes place, and of the evolution of social norms. – Today game theory is integrated into mainstream economic reasoning. Repeated Games and Game Theory. If games are repeated then there is the possibility of punishing people for cheating, this will provide an incentive for sticking to the Pareto optimal approach. However, if they are repeated a finite number of times then there will be an incentive to cheat. If the game is played 10 times then the player will defect on the 10th round so why cooperate. So, therefore, you may as well defect on round 9 and so round 8 as well. – Game Theory: A game of entry deterrence. If a new firm enters the market then the payoff will depend on whether the incumbent fights or accepts. If the incumbent fights they both get 0. If it does not fight then the incumbent gets 1 and the entrant gets 2. Therefore the equilibrium is for the new firm to enter and the incumbent to accept.