Mathematical Economics at Cowles[1]

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Introduction

The first fifty years of Cowles were a period of profound and far-reaching transformation in the field of mathematical economics. A few numbers help one to perceive the depth and the extent of the change that took place in the half-century just ended. Both *Econometrica* and *The Review of Economic Studies* began publication in 1933. The first volume of the former for the calendar year 1933 and the first volume of the latter for the academic year 1933–34, together, devoted 165 pages to articles contributing to mathematical economics, by a liberal definition of terms. According to a stricter definition, the corresponding number of pages for the year 1982 was close to 4,000 for the five main periodicals in the field (*Econometrica, Review of Economic Studies, International Economic Review*, *Journal of Economic Theory*, and *Journal of Mathematical Economics*). No less remarkable was the metamorphosis of the *American Economic Review*. Its entire volume for the year 1933 contained exactly four pages where any mathematical symbol appeared, and two of them were in the Book Review Section. In contrast, in 1982 only five of the fifty-two articles published by the *A.E.R.* were free from mathematical expressions. Still another index of the rapid growth of mathematical economics during that period can be found in the increase in membership of the Econometric Society, which went from 163 on January 1, 1932 to 2,987 in 1982.

Having heeded the demanding motto "Science is Measurement," under which the Cowles Commission worked for its first twenty years, we may look at non-quantifiable aspects of the development of mathematical economics from the early thirties to the present. Of these, one of the most impressive has been the steadily increasing sophistication of the mathematical tools used by economic theorists. To differential calculus and matrix algebra were gradually added, in a nonexhaustive list, convex analysis, set theory, general topology and algebraic topology, measure theory, infinite-dimensional vector space theory, global analysis, and nonstandard analysis. Simultaneously, as economic theory was axiomatized, exacting standards of logical rigor became the rule rather than the exception. The sweep of the recent evolution of mathematical economics may tempt one to find in that historical process an inevitable concatenation of events and to overlook the major accidents that altered its direction or increased its momentum. But historical determinism does not fully explain John von Neumann's interest in game theory and in economics and the circumstances that led to his collaboration with Oskar Morgenstern.

Nor does it explain the role played by Alfred Cowles. Carl F. Christ has recounted *in Economic Theory and Measurement* Irving Fisher's delight and Charles F. Roos's incredulity at an investment counselor in Colorado Springs offering to be the Maecenas of a group for fundamental

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[1] Among the historical sources from which this paper drew, one of the most valuable was the Cowles Reports covering periods varying from one year to twenty years, and especially Carl Christ's contribution to the *Report for 1932–1952*. I am grateful to K. J. Arrow, K. Borch, L. Hurwicz, L. W. McKenzie, A. Mas-Colell, R. Radner, A. H. Taub, and J. Tobin for their comments, and to the National Science Foundation for its support.

economic research. How much greater would the delight and the incredulity have been, had Fisher and Roos anticipated how successful the group and how fundamental the research would become. In mathematical economics alone, the Cowles Commission and the Cowles Foundation were to make major contributions to several of the main developments over the next five decades. And during that period virtually every important new idea in economic theory found a rich resonance at Cowles. To illustrate these assertions I will review a history that unfolded over a semicentury in the canonical microcentury of von Neumann, a time contraction that will permit only too brief a spotlighting of too few events.

Colorado Springs (1932–1939)

Even with hindsight one cannot easily decipher the genetic code of Cowles shortly after its conception. From 1932 until the fall of 1934, the research staff was composed of Alfred Cowles, Forrest Danson, Harold T. Davis, and William F.C. Nelson. Their work was oriented toward Alfred Cowles's concern with the analysis of stock market prices and, more generally, toward the applications of statistical methods to economics.

The first Cowles Commission monograph, *Dynamic Economics*, was extremely mathematical for 1934, the year in which it was published. It is not a genuine Cowles product, however, since it was completed by the time its author, Roos, joined the Commission and became its research director in September 1934. Two other mathematical economists were later added to the staff but stayed only for short periods. Gerhard Tintner was a research fellow for a little less than a year in 1936–37, and Abraham Wald, who was appointed as a research fellow for one year in July 1938, actually left for Columbia University in September 1938. The deep, original work that Wald had done in Vienna on the existence of a competitive equilibrium was already a few years old. At the time of his brief visit to Colorado Springs his research in economic theory concerned work that would later be reported in approximate Determination of Indifference Surfaces by Means of Engel Curves" and "A New Formula for the Index of Cost of Living," which were published in *Econometrica*, April 1940 and October 1939, respectively.

Although the list of mathematical economists on the research staff of the Cowles Commission from 1932 to 1939 was short, the roster of participants in the summer conferences held in Colorado Springs from 1935 to 1940 was long. It included many leading economic theorists who came from Western Europe as well as from the United States.

The scenic beauty of the Colorado Springs area contributed to the success of the six summer conferences that the Cowles Commission hosted but could not compensate for its remoteness. After Roos's resignation in January 1937, the difficulty of attracting a new research director insistently suggested a move. Christ reports that Northwestern, the University of California at Los Angeles, and Yale expressed interest and that Northwestern became the leading contender when Alfred Cowles decided to return to Chicago in 1939. It was to the University of Chicago, however, that the Cowles Commission eventually moved in September 1939. The move to Yale would have to wait for sixteen years.

In 1939 the Cowles Commission had reached a critical point, and we can, for a nanocentury, indulge in speculating about the other branches it might have followed, in particular, about the branch that would have led us to gather today in Evanston as the guests of Robert Strotz.

Chicago (1939–1955)

The move to Chicago was associated with the appointment of a new research director, Theodore O. Yntema, and several additions to the research staff, notably Oscar Lange and Jacob L. Mosak, who contributed respectively the seventh and the eighth monographs to the Cowles Commission series. Mosak's *General Equilibrium Theory in International Trade* (1944) was based on the doctoral dissertation he completed at the University of Chicago in 1941. Some of the ideas of *Lange's Price Flexibility and Employment* (1944) had been presented at the New Orleans meeting of the Econometric Society on December 28, 1940. Both books are in the tradition of the theory of general economic equilibrium, which remains to the present day one of the main research themes at Cowles and which was thus already alive at the Commission soon after its move to Chicago. There is a significant difference in style between the two volumes. While Mosak used mathematics freely in his text, Lange confined it to footnotes and to an appendix, an example that would not be followed by any of the later Cowles monographs on economic theory. Mosak's and Lange's books commanded professional attention, and they showed that Cowles had found a vocation for mathematical economics that it would not lose.

During the first two years of the Cowles Commission in Chicago, Lange was also working on "The Foundations of Welfare Economics," a paper he presented on September 4, 1941 at the Sixth International Congress for the Unity of Science and published in *Econometrica* for July–October 1942. In this article, Lange studied the characterization of the Pareto optima of an economic system by means of differential calculus, a problem that Maurice Allais was independently considering at about the same time in France in his *A la Recherche d'une Discipline Economique*, published in 1943. Lange's and Allais's contributions brought a long phase in the development of the two basic theorems of welfare economics close to its conclusion. They were to influence the reexamination of those two theorems by means of convex analysis in the early fifties.

The entry of the United States into World War II had an instant, direct impact and a far greater delayed, indirect impact on the work of the Cowles Commission. A study of price controls was undertaken, and Leonid Hurwicz, who had joined the research staff in July 1941, was initially scheduled to devote full time to the project after mid-1942 with a team of interviewers, in an early encounter with the design of mechanisms for resource allocation.

During the next two years, with uncanny judgment, the Cowles group coopted three new members. Jacob Marschak became research director in January 1943, beginning a deeply interactive association with Cowles that would last for more than 17 years. Tjalling C. Koopmans joined as a research associate in July 1944, forming with Cowles a bond that has not been broken in the 39 years since then. His appointment was followed in November 1944 by that of Lawrence R. Klein, who left the Commission in July 1947 after a shorter but no less intense association.

1944 was also the year in which the *Theory of Games and Economic Behavior* appeared. Not all members of the profession were unaware that a golden age was dawning for mathematical economics. Both Hurwicz, in the *American Economic Review* for December 1945, and Marschak, in the *Journal of Political Economy* for April 1946, wrote long, perceptive review articles, in which Hurwicz recognized that “a rare event” had taken place and Marschak that he had read an “exceptional book.” However, the six years that followed the first edition of the work of von Neumann and Morgenstern disappointed the expectations of economists. Most of the research done on the theory of games at that time focused on the MinMax Theorem, which eventually moved off center stage and now plays a minor supporting role. Only after 1950, in particular after the publication of John Nash’s one-page article on the existence of equilibrium points in *N*-person games in the *Proceedings of the National Academy of Sciences* for January 1950, did the theory start again to acquire the dominant features that characterize it today. Ever since 1944 the Cowles group had kept abreast of the evolution of the theory of games by, for instance, inviting von Neumann to give two lectures on May 25 and 26, 1945, and organizing a series of seven seminars on the subject given from January to April 1949 by Leonard J. Savage, Kenneth J. Arrow, Marschak, Meyer A. Girschik, and Herbert A. Simon. Thus the group was permanently exposed to the influence of the various developments of the theory. Yet it had to wait until 1963, when Martin Shubik joined the Cowles Foundation, to have a professional game theorist in residence.

In the mid- and late-forties, the major share of the research effort at the Cowles Commission was spent on work reported in *Statistical Inference in Dynamic Economic Models*, which appeared in 1950 under Koopmans’s editorship. But mathematical economics was far from being neglected. Koopmans was already at work in 1945 on production theory in the spirit later embodied in *Activity Analysis of Production and Allocation* (1951), of which he was also the editor. Arrow, who was a member of the commission from April 1947 to June 1949, had begun at RAND in the summer of 1948 a study of the aggregation of individual value judgments that would result in his *Social Choice and Individual Values* (1951), the twelfth of the Cowles monographs. The impossibility of such an aggregation under the conditions specified by Arrow has fascinated a large number of social scientists for more than three decades, and it has inspired a vast literature that is still growing rapidly.

The close of the forties brought together at the Cowles Commission an interdisciplinary group of some fifty research workers for a conference held on June 20–24, 1949, the proceedings of which were published as the thirteenth Cowles monograph, *Activity Analysis of Production and Allocation*. Koopmans contributed the Introduction as the editor and a major chapter, “Analysis of Production as an Efficient Combination of Activities,” and he authored or coauthored three other chapters. The monograph also contained the classic paper of George B. Dantzig presenting the simplex algorithm for the constructive solution of linear programming problems. With the passage of time, that conference has stood out more and more clearly as an important moment in the history of mathematical economics. The theory of production was looked at from new viewpoints; the computation of optimal production programs received emphasis; convex analysis was developed for the needs of production theorists and extensively applied; the observance of mathematical rigor was taken for granted; and another demonstration of the fecundity of interaction among economists, mathematicians, and operations researchers was given.

In the last years of the forties, the theory of utility had become a focus of interest at the Cowles Commission. In one of the seven seminars on the theory of games given from January to April 1949, Marschak had chosen to speak on the theory of measurable utility that von Neumann and

Morgenstern had obtained as a byproduct of their work. That theory, which became fundamental to the study of decision making under uncertainty, was in need of the significant clarification that Marschak provided in his article in *Econometrica* for April 1950. Approximately at the same time Herman Rubin, who was a Cowles research associate in 1946–47 and in 1948–49, wrote, but never published, “An Axiom System for Measurable Utility,” in which he extended the theory of measurable utility to the case of an infinity of sure prospects. Later, another substantially different and notably simpler extension in the same direction was the subject of the “Axiomatic Approach to Measurable Utility,” in *Econometrica* for April 1953, by I.N. Herstein of the Cowles Commission and John Milnor of Princeton University: The work done at the Commission on these questions invited a natural symbiosis with the research on personal probability that Savage carried out in the Committee on Statistics of the University of Chicago, which would appear *as The Foundations of Statistics* in 1954. An exchange of ideas took place within that channel too.

Two different areas of the theory of preferences, outside the framework of von Neumann and Morgenstern, were also explored at that time.

Koopmans was attempting to give a formal theory of the desirability that an economic agent postpone decisions. This was the subject of a paper he presented at the Boulder meeting of the Econometric Society on August 29, 1949 and of the article, “On Flexibility of Future Preference,” which he published fifteen years later. The representation of preferences by a utility function in the case studied by von Neumann and Morgenstern makes essential use of the algebraic structure introduced by the formation of probability mixtures of prospects. The more basic question of whether a preference preorder on a topological space can be represented by a real-valued utility function was the object of a paper published in 1954 by Debreu, who became a Cowles research associate in June 1950 for an eleven-year term.

With the fifties began a new phase in the development of the theory of general economic equilibrium, which was due for a thorough reexamination. The first problem to be investigated was the characterization of the Pareto optima of an economy by means of a price system. The many contributions made to this subject since the end of the nineteenth century had relied on differential calculus, which made it difficult to handle consumption vectors in the boundary of consumption sets or production sets with nonsmooth boundaries. As a consequence, those contributions did not satisfy the new, more exacting standards of rigor prevailing in economic theory. It now turned out that the traditional, troublesome assumptions of differentiability were superfluous, and that convex analysis, in particular the supporting hyperplane theorem, provided a rigorous answer that was more general, more natural, and simpler.

This was done in two papers presented independently and almost simultaneously in the summer of 1950 in Berkeley by Arrow (then on the Stanford faculty) and at Harvard by Debreu. Their results, published in 1951, were later extended by Debreu (*Proceedings of the National Academy of Sciences* for July 1954) to infinite-dimensional commodity spaces. Such spaces are called for when the time horizon of the economy is infinite; or time, location, or quality is treated as continuous; or the set of states of the world is infinite. Economies with an infinite time horizon had already been the subject of Edmond Malinvaud’s research while he was a guest of the Cowles Commission in 1950–51. His article in *Econometrica* for April 1953 characterized efficient programs of capital accumulation for such economies, again by means of convex analysis.

The second, more fundamental problem to be attacked was the existence of a competitive equilibrium. Walras and his disciples were aware of the need for arguments in support of the existence of the central concept of his theory. But for more than half a century the observation that the number of equations in his model equaled the number of unknowns was the only, although insufficient, reason given. An extraordinary set of circumstances led to the first solution of the existence problem by Wald in Vienna in the early thirties. One of Wald's papers on this subject was lost, two were published in the *Ergebnisse eines mathematischen Kolloquiums* (for 1935 and 1936), and the last appeared in the *Zeitschrift für Nationalökonomie* for December 1936. The obscurity of the *Ergebnisse*, the novelty and the difficulty of Wald's mathematical arguments, as well as the fact that he wrote in German, help to explain why no other work was done on the problem of existence of a competitive equilibrium for almost twenty years, and why his contribution received little attention until an English translation of the *Zeitschrift für Nationalokonomie* article was published in *Econometrica* for October 1951.

Actually, the paper that von Neumann wrote for the 1937 *Ergebnisse* on the existence of optimal balanced-growth paths was far more influential in the period of the subject's development that was about to begin. Von Neumann's paper, which appeared in English translation *in The Review of Economic Studies* for 1945–46, contained a lemma that was reformulated as a fixed-point theorem for correspondences by Shizuo Kakutani in the *Duke Mathematical Journal* for September 1941. Nash's article of 1950, on the existence of equilibrium points in *N*-person games, and Morton L. Slater's Cowles Commission Discussion Paper "Lagrange Multipliers Revisited," also of 1950, introduced social scientists to Kakutani's result, which soon became the most powerful tool for proofs of existence of equilibrium for social systems.

Thus, Arrow at Stanford University and Debreu at the Cowles Commission, separately at first, jointly later, used Kakutani's theorem to establish the existence of a competitive equilibrium for an economy made up of consumers and producers interacting through markets for commodities. Their results gave rise to an article by Debreu (*Proceedings of the National Academy of Sciences* for October 1952) devoted to the existence of a generalized Cournot-Nash equilibrium for an abstract social system, and to a paper by Arrow and Debreu presented on December 27 at the 1952 meeting of the Econometric Society in Chicago and published in *Econometrica* for July 1954. At the same time, at Duke University, Lionel W. McKenzie used Kakutani's theorem to prove the existence of an "...Equilibrium in Graham's Model of World Trade and Other Competitive Systems," in a paper that he gave on December 28 at the 1952 meeting of the Econometric Society in Chicago and published in *Econometrica* for April 1954.

Toward the end of the Chicago years of the Cowles Commission it became apparent that the complex proofs of Arrow and Debreu could be simplified if one replaced their approach, in which all the agents (including a fictitious market agent) simultaneously try to optimize their objective functions, by a more classical approach. Specifically, let us define the excess demand correspondence of an economy by associating with every price vector in a suitable domain the set of excess demand vectors in the commodity space to which that price vector gives rise. The proof of existence of a competitive equilibrium for an economy can be based on a lemma asserting the existence of a price vector for which the associated excess demand set intersects the closed negative cone of the commodity space. This lemma was independently obtained, as a direct application of Kakutani's theorem, at the University of Copenhagen by David *Gale (Mathematica*

Scandinavica, 1955), at the Tokyo College of Science by Hukukane Nikaidô (*Metroeconomica*, August 1956), and at Cowles by Debreu, who published it in a generalized form in the *Proceedings of the National Academy of Sciences* for November 1956. The papers that Wald wrote in the early thirties had received no echo for nearly two decades, but in the early fifties the time of proofs of existence of a general economic equilibrium had clearly come.

Economists had long been familiar with the idea that a commodity is defined by its physical characteristics and by the date and the location at which it is available. At a colloquium of the Centre National de la Recherche Scientifique held in Paris on May 12–17, 1952, Arrow suggested that a new theory of economic uncertainty could be obtained by adding to the characteristics of a commodity the state of the world in which it is available. His article, “Le rôle des valeurs boursieres pour la repartition la meilleure des risques” was published in *Econometrie*, 1953 and, in English translation, in *The Review of Economic Studies* for April 1964. Extensions of this idea were the subject of a paper written in the summer of 1953 by Debreu, then at Electricité de France on leave from the Cowles Commission, and published in 1959 as Chapter 7 of his *Theory of Value*, the seventeenth Cowles monograph. In a textbook application of the axiomatic method, the redefinition of one of the primitive concepts of the theory of general economic equilibrium, the concept of a commodity, had provided a theory of uncertainty, without any change in the form of the model.

In a category by itself was the study of “Optimal Inventory Policy” by Arrow, Harris, and Marschak in *Econometrica* for July 1951, which determined optimal rules of stock management in the face of stochastic demand, the cost of placing orders, and penalties for depletion. Their article stimulated a substantial body of work, some of which was done later at the Cowles Foundation.

Absorbing as the problems mentioned so far were, they represented only a fraction of the research effort that the Cowles Commission applied to mathematical economics from the mid-forties to the mid-fifties. To appreciate the intensity of that effort, one must read the list of the additional staff appointments made in that area during that period.

Don Patinkin was a member of the Cowles Commission from May 1946 to June 1948; Evsey D. Domar, from July 1947 to July 1948; Stanley Reiter, in the summer of 1948 and from March 1949 to September 1950. Franco Modigliani served as a research associate from September 1948 to November 1948, and Clifford Hildreth from January 1949 to September 1953. Harry Markowitz joined the staff in October 1949; Roy Radner in March 1951; Martin J. Beckmann in July 1951; Hendrik S. Houthakker and C.B. McGuire in January 1952; Christopher B. Winsten in October 1952; Arnold C. Harberger in October 1953; and Richard Muth in April 1954. Hurwicz, who had left the Commission in mid-1946, became a research associate again from October 1950 to September 1951.

The list is remarkable, but it does not yet convey an accurate impression, for it leaves out the numerous guests and consultants who participated actively in the scientific life of the Cowles Commission during its last decade in Chicago. The questions of economic theory on which the members of that group worked covered a wide range that included monetary theory, the theory of portfolio selection, the theory of location and of transportation, the theory of international trade, and the theory of organizations, among others. These questions eventually gave rise to two additional Cowles monographs.

Markowitz had undertaken a study of the problem of *Portfolio Selection* based on the theories of measurable utility and of personal probability, which were the subject of active research in Chicago. Central to his analysis was the concept of an efficient portfolio-a portfolio that is not dominated by any other in terms of higher expectation of return and lower variance of return. Markowitz's influential work appeared in 1959 as the sixteenth Cowles monograph. In this case the research interests of the Commission seemed to have gone full circle and to have returned to Alfred Cowles's original concern with investment counseling, but the object of Markowitz's book was essentially different.

Shortly after Radner became a member of the Cowles Commission, he began his collaboration with Marschak on the *Economic Theory of Teams* that extended over two decades. Marschak and Radner defined a team as an organization whose members have the same interests and beliefs, but who do not have the same information. Their work centered on problems concerning the optimal flow of information within the team. It was published in 1972 as the twenty-second Cowles monograph. Several of the questions studied by Marschak and Radner are basic to any theory of organizations. They belong to the broad area of the economics of information, which is still, and is likely to remain, under intensive exploration by economic theorists.

In the summer of 1955 the Chicago period of Cowles came to an end. Intellectual excitement had been sustained at a high level in staff meetings and seminars, through personal interaction, and through the circulation of more than 350 Discussion Papers. Several research projects had been completed and others were under way. The many staff members, guests, and consultants of the Commission during that period must share the feeling of having participated at a privileged time in an exceptional adventure.

New Haven (1955–1982)

Five members of the staff (Beckmann, Debreu, Koopmans, Marschak, and Radner) took part in the move of the Cowles Commission from Chicago to New Haven, where they merged with Harold W. Guthrie, Robert Summers, and James Tobin of Yale University. In the merger, the commission became a foundation under Tobin's directorship.

In perfect contrast with Roos's *Dynamic Economics*, the first book published by a member of the Cowles Foundation was a genuine Cowles product but was not a Cowles monograph. In the first of his *Three Essays on the State of Economic Science* (1957), Koopmans presented the recently reexamined theories of consumption, production, and general economic equilibrium. The other two essays were concerned with broader issues of methodology in economic theory. The axiomatic approach and the introduction of more fundamental mathematics were assessed positively. After a probing checkup, mathematical economics was found to be in good health. Indeed, its vitality was to manifest itself soon in new directions.

One of them was the revival of interest in the problem of stability of competitive equilibrium in the framework defined by Paul A. Samuelson in *Econometrica* for April 1941. In a major contribution to that revival, Arrow and Hurwicz had written in *Econometrica* for October 1958: "The nature of our findings can be summarized very simply by saying that in none of the cases studied have we found the system to be unstable under the (perfectly competitive) adjustment process, whether instantaneous or lagged." Thus, Herbert E. Scarf had a ready professional audience when, in work done partly during his 1959–60 visit to the Cowles Foundation, he constructed a class of globally unstable economies. In his article in the *International Economic Review* for September 1960, Scarf concluded "it seems clear that instability is a common phenomenon," an assertion fully confirmed by the characterization of excess demand functions in the early seventies.

In another direction, a closer bond was created between economic theory and game theory. In 1881, in his *Mathematical Psychics*, Edgeworth had shown how the "contract curve" of an economy with two commodities and two consumers tends to the set of its competitive allocations as one replicates the economy in an increasingly larger number of copies. In spite of its brilliance, Edgeworth's contribution received little notice for almost eighty years. It was Shubik who related the contract curve to the game-theoretical concept of the core, in his article of 1959 in *Contributions to the Theory of Games IV*, and sparked the research of the sixties and seventies on the core of an economy. The first generalization of Edgeworth's result was presented by Scarf at a meeting held in Princeton on October 4–6, 1961, whose proceedings were circulated as *Recent Advances in Game Theory*, 1962. Further work on this question was done by Debreu during his visit to Cowles in the fall of 1961, and later at Berkeley, and by Scarf; first at Stanford, then at Cowles (which he joined in July 1963). It led to their paper, "A Limit Theorem on the Core of an Economy," in the *International Economic Review* for September 1963, which proved the asymptotic equality of the core and the set of competitive allocations for replicated economies with arbitrary numbers of commodities and consumers.

In yet another direction, game theory provided a decisive stimulus to economic theory. The original proof of existence of a pair of equilibrium strategies for two-person zero-sum games given by von Neumann in 1928 used complex topological arguments. Later derivations of the MinMax Theorem reduced it to an application of the supporting hyperplane theorem for convex sets, and the theory of linear programming provided algorithms for the construction of solutions. Thus, after Nash's proof of the existence of an equilibrium for an *N*-person game in 1950, the question arose whether in the case of a two-person non-zero-sum game, one could also develop an algorithm for the construction of a pair of equilibrium strategies. The positive answer to that question given by C.E. Lemke and J.T. Howson in *SIAM Journal of Applied Mathematics*, June 1964, was the starting point of Scarf's work on the nonemptiness of the core of a balanced *N*-person game without side-payments (*Econometrica*, January 1967) and of his work on fixed-point algorithms leading to publication of *The Computation of Economic Equilibria*. The latter title was the twenty-fourth Cowles monograph, which Scarf published in 1973 with the collaboration of Terje Hansen. In a major advance, efficient algorithms were now available to compute an approximate general economic equilibrium, and their applications to a variety of fields of economics have proliferated during the last decade.

The theory of fixed-point algorithms was later set in a more general framework that included the theory of complementary pivot algorithms by B. Curtis Eaves, a Cowles visitor during the academic year 1974-75, and Scarf in their article, "The Solution of Systems of Piecewise Linear

Equations,” in *Mathematics of Operations Research* for February 1976. In the fall of 1974 Stephen Smale was also visiting the Cowles Foundation. The research he did at that time on Global Newton Methods, which he published in the *Journal of Mathematical Economics* for July 1976, paralleled, in a differential context, the work of Eaves and Scarf. The interaction that developed then among Eaves, Scarf, and Smale had a delayed influence on the breakthrough that Smale later made in his study of the average speed of convergence of the simplex method. The convenient form in which Eaves and Scarf had recast the linear complementarity problem provided one of the strands that Smale laid in his work, begun in the fall of 1981. Still in the area of algorithms motivated by economic theory, Scarf started in 1977 an investigation of challenging problems of integer programming suggested by the theory of production with indivisibilities. It gave rise to his two articles in *Econometrica* for January 1981 and March 1981.

The theory of the core of a large economy had led Robert J. Aumann to introduce atomless measure spaces of economic agents to formalize the concept of a nonnegligible set of negligible agents, in *Econometrica* for January-April 1964. An alternative formalization was proposed by Donald J. Brown and Abraham Robinson in the *Proceedings of the National Academy of Sciences* for May 1972. Its basis was the theory of infinitesimals developed in the early sixties by Robinson, who became a member of the Yale faculty in the fall of 1967. His collaboration with Brown, who joined the Cowles Foundation in September 1969, introduced the sophisticated techniques of nonstandard analysis into economic theory, and it gave the initial impulse to the study of the theory of general economic equilibrium in this new mathematical framework, a research program in which Brown was a leader. Nonstandard analysis also revealed itself to be an effective tool for the discovery of standard results and proofs in economic theory. The “Elementary Core Equivalence Theorem,” published by Robert M. Anderson in *Econometrica* for November 1978, is indeed elementary in its statement as well as in its proof. Yet its motivation is to be found in the doctoral dissertation on nonstandard analysis that he wrote at Yale in 1977. The apparition in economic theory of Robinson’s unfamiliar ideas gave mathematical economists an exceptional opportunity to scrutinize their defense mechanisms in the presence of a foreign body. Once again the healthy reaction of the profession was not one of rejection.

In New Haven, as in Chicago, the theory of utility received frequent attention. One of the topics in the research done in this field was the theory of stochastic choice, in which an agent presented with a set of alternative actions chooses one of them according to a probability distribution. Marschak, who played the leading role at Cowles in this area, published five articles on this subject from 1955 to 1960 (two of them with H.D. Block and one with Donald Davidson), the last of which appeared in *Mathematical Methods in the Social Sciences*) 1959, edited by Arrow, Karlin, and Suppes. To this program Debreu contributed four papers published from 1958 to 1960, the last of which (also in the Arrow–Karlin–Suppes volume) characterized preferences defined on a finite product of consumption sets and representable by an additively decomposed utility function.

Another topic recurring in the research of the Cowles Foundation on utility theory was Böhm-Bawerkian impatience, which appeared in several, sometimes barely recognizable, forms. It was the subject of two articles in *Econometrica*, one by Koopmans in April 1960 and one by Koopmans, Peter A. Diamond, and Richard E. Williamson in January-April 1964; and of two chapters by Koopmans in *Decision and Organization*, a volume published in honor of Marschak in 1972 by McGuire and Radner. These contributions to the theory of impatience called on properties of various topologies on infinite-dimensional commodity spaces. The study of these properties was pursued further by Brown and Lucinda M. Lewis in their article, “Myopic Economic Agents,” in *Econometrica* for March 1981. Of particular interest to Brown and Lewis was the Mackey topology, which had been introduced in economics in 1970 by Truman F. Bewley, a future member of Cowles, in his first doctoral dissertation.

Partially motivated by the consideration of additively decomposed utility functions, Koopmans and Menahem E. Yaari had separately begun an investigation of additively decomposed quasiconvex functions in the early seventies. This work eventually gave rise to an article by Yaari in *Econometrica* for July 1977, and, after a visit by Debreu to the Cowles Foundation in the fall of 1976, to an article by Debreu and Koopmans in *Mathematical Programming* for September 1982.

The theory of revealed preference originated by Samuelson (*Economica*, February and August 1938 and November 1948; *Econometrica*, October 1938; *Foundations of Economic Analysis*, 1947) provided a further instance of work on utility theory in New Haven. The theory had made a great leap forward with the strong axiom that Houthakker presented in *Economica* for May 1950. In research done in part during a visit to Cowles in 1964-65, Sidney N. Afriat reexamined that axiom from a different viewpoint and published his results in “The Construction of Utility Functions from Expenditure Data” (*International Economic Review*, February 1967).

The sixties at Cowles saw a burst of research activity on the theory of growth that resulted in some twenty-five articles. One of the main themes, originating in von Neumann’s paper in the 1937 *Ergebnisse eines mathematischen Kolloquiums*, was balanced growth. A first variation on that theme studied the Golden Age whose lineage can be traced back to James Meade’s *Introduction to Economic Analysis and Policy*, 1936, Chapter 3, since balanced growth is formally equivalent to stationarity. An explicit mathematical treatment appeared in Allais’s *Economie et Intérêt*, 1947, Chapter 7, and a solution by means of convex analysis in Malinvaud’s article in *Econometrica*, April 1953, Section 20. At the beginning of the sixties, the topic was given prominence by the work of Edmund Phelps at Cowles (*American Economic Review*, September 1961, December 1962, September 1965), and of several authors elsewhere. A second variation studied the turnpike property of balanced growth discovered by Dorfman, Samuelson, and Solow in *Linear Programming and Economic Analysis*, 1958, Chapter 12, which provided a powerful stimulant for economic research in the following decade. During the same period the area of the theory of growth straddling the boundary between microeconomics and macroeconomics was explored in contributions that are surveyed by Robert M. Solow in “Cowles and the Tradition of Macroeconomics” in this volume. In addition to Phelps, Beckmann, David Cass, Emmanuel M. Drandakis, Koopmans, T.N. Srinivasan, Joseph E. Stiglitz, Tobin, Martin L. Weitzman, and Yaari generated and sustained the momentum of research on growth at Cowles at that time.

Inventory theory, strongly stimulated in the early fifties by the article of Arrow, Harris, and Marschak, remained a focal point of interest at the Cowles Foundation, and Beckmann, Koopmans, and Richard F. Muth were among its main contributors. The adjacent fields of economic theory, operations research, linear programming, and dynamic programming provided many opportunities for cross-fertilization illustrated in the late fifties by the work of Alan S. Manne.

The theory of games, which in 1963 had already influenced economic theory so deeply and in so many ways, became in its own right a subject of intensive research at the Cowles Foundation after Shubik's appointment twenty years ago. It may indeed seem to a speaker at some distant Cowles anniversary, especially if he reads the 37-part Discussion Paper on "The Theory of Money and Financial Institutions," that Shubik attempted to create the myth that he did not exist and that his was a pen name for an anonymous team of game theorists. Having linked the contract curve and the core earlier, Shubik now built another important connection between the theory of games and the theory of general economic equilibrium in the theory of strategic market games. In these various projects, one of the closest and most frequent of his collaborators was Pradeep Dubey, who joined the Cowles Foundation in September 1975.

By the end of 1982 the Cowles Foundation had released more than 650 Discussion Papers, and the Cowles Discussion Paper Series, created in 1947, had passed the 1,000 mark. The mere reading of the titles and of the authors' names in the areas of mathematical economics and of economic theory would require several microcenturies. Therefore, whole research programs of the Cowles Foundation will only be alluded to in a list that will be even less complete than for the Chicago years. Research begun at the Commission was continued at the Foundation on the theory of portfolio selection, on the theory of teams and of organizations, on the economics of information, on location theory, on the assignment problem, and on social choice. Monetary theory, oligopoly, the theory of risk, foreign trade and development were the subjects of programs started in New Haven, and among the more recently initiated projects are the theory of planning, the study of increasing returns, and the study of excess demand functions.

Conclusion

To an observer of mathematical economics in the last half-century, progress from one year to the next was often only just noticeable. The change from any decade to the next was significant, however, and the transformation over the entire fifty-year interval was striking. This seems to confirm that now is the proper time to pause for a special assessment of the contributions of Cowles to mathematical economics.

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Fundamental Methods of Mathematical Economics. Kevin Wainwright. 3.9 out of 5 stars 54.Â Lawrence Blume is professor of economics at Cornell University. He received his Ph.D. from the University of California, Berkeley, and has taught at Harvard Universityâ€™s Kennedy School, the University of Michigan, and the University of Tel Aviv. Product details. Publisher : W. W. Norton & Company (April 17, 1994).Â If you're starting graduate studies in economics, then I'd also strongly recommend Sundaram's A First Course in Optimization Theory . Its subject matter is mostly a subset of Mathematics for Economists, but optimization an often-tricky topic that you'll need to master. Read more. The introduction of mathematical methods into economics has been a long process. Pieces of essentially mathematical reasoning applied to economic problems have been detected as far back in history as in Aristotleâ€™s work (see Theocaris (1961)) and in the XVIII and early XIX centuries, outstanding mathematicians such as Bernoulli, Gauss, Laplace and Poisson developed truly mathematical models to discuss economic problems. But the rise of mathematical economics in the modern sense is usually dated back to Cournotâ€™s (1838) classical (and long neglected) research on microeconomic theory.Â The Theory of Value: An axiomatic analysis of economic equilibrium, Cowles Foundation Monographs Series. Methods of Mathematical Economics Instructorâ€™s Manual 2. (a) The endogenous variables are Y, C, and G. (b) g = G/Y = proportion of national income spent as government expenditure. (c) Substituting the last two equations into the first, we get Y = a + b(Y âˆ' T0) + I0 + gY Thus a âˆ' bT0 + I0 Yâˆ— = 1âˆ'bâˆ'g (d) The restriction.Â 3. Upon substitution, the first equation can be reduced to the form Y âˆ' 6Y 1/2 âˆ' 55 = 0 or w2 âˆ' 6w âˆ' 55 = 0 (where w = Y 1/2) The latter is a quadratic equation, with roots âˆ™ Â¸ 1 w1âˆ— , w2âˆ— = 6 Â± (36 + 220)1/2 = 11, âˆ'5 2 From the first root, we can get. Y âˆ— = w1âˆ—2