

On irreducible spherical tensor operators and “modern” textbooks on quantum mechanics: the perpetuation of nonsense.

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Despite its title, this note does not attempt a case study nor a serious sociological, or philosophical discussion (c.f. [2]). It just presents an instance of a phenomenon which the title attempts to describe; specifically how two modern textbooks present the subject ‘irreducible spherical tensor operators’ in quantum theory. It was written for the students taking the 2014 edition of the lectures in Quantum Mechanics at the Facultad de Matemática, Astronomía y Física of the national university in Córdoba, Argentina; and, with minor expectations, for some of my colleagues. To inform the reader on my perspective, I must say that upon returning to the teaching of quantum mechanics after spending roughly ten years teaching other subjects in our physics curricula, I found a certain ‘modernization’ had taken place. This was, for example, apparent in the exercises where I detected a persistent presence of notations and formulations used by texts such as the well-known books by Ballentine [1] or that by Zettili [8]. I must also confess that I hadn’t read these texts at all. If sorely pressed by some serious student who asks me to recommend a quantum mechanics textbook, my answer is, invariably, the book by Albert Messiah, [4], the preface of which is dated October, 1958. After my lectures on irreducible spherical tensor operators (I will shorten this to ISTO’s) a student, Natalia Giovenale, informed me that my story was in blatant contradiction with Zettili’s presentation. So I had to do my chores and here is what I saw.

In Ballentine’s book ISTO’s are presented in section 7.8 beginning on page 193; there you are informed that a set of $2k + 1$ operators $\{T_q^{(k)} : (-k \leq q \leq k)\}$ form an *irreducible tensor of degree k* ¹ if

$$(1) \quad \mathbf{R}(\alpha, \beta, \gamma) T_q^{(k)} \mathbf{R}^{-1}(\alpha, \beta, \gamma) = \sum_{q'} T_{q'}^{(k)} D_{q',q}^{(k)}(\alpha, \beta, \gamma) .$$

This is eq. (7.108) of Ballentine and is Wigner’s original definition [7]. With apologies to the knowledgeable, the various symbols stand for the following. (α, β, γ) are the Euler angles for a rotation in real three dimensional euclidean space around a direction $\hat{\mathbf{n}}$ (in \mathbb{R}^3 with unit length) by an angle ϕ , say with the usual convention regarding the sign of ϕ and the sense of rotation.

$$(2) \quad \mathbf{R}(\alpha, \beta, \gamma) = \exp\{-i\phi\hat{\mathbf{n}} \cdot \mathbf{J}/\hbar\} ,$$

¹The qualifier ‘spherical’ is not used which is perfectly alright although I will continue to insist with the acronym ISTO.

where the angular momentum operator-triple $\mathbf{J} = (J_1, J_2, J_3)$ consists of self-adjoint operators acting on a Hilbert space and satisfying the commutation relation $[J_1, J_2] = i\hbar J_3$ and the other two obtained from this one by cyclic permutation of the indices $\{1, 2, 3\}$ ². The $(2k+1) \times (2k+1)$ matrix $D(\dots)$ whose matrix elements (in an orthonormal basis to be specified shortly) enter in the right-hand side of Wigner's defining equation (1) is

$$(3) \quad D(\alpha, \beta, \gamma) = \exp\{-i\phi\hat{\mathbf{n}} \cdot \mathbf{S}^{[k]}/\hbar\},$$

where the angular momentum $\mathbf{S}^{[k]}$ is an irreducible angular momentum of magnitude k ; that is to say it is an angular momentum \mathbf{S} acting on a Hilbert space (say \mathcal{K}) in such a way that $[A, S_j] = 0$ for $j = 1, 2, 3$ and for an operator A on \mathcal{K} implies that A is a multiple of the identity³. It follows, because \mathbf{S}^2 always⁴ commutes with each component of \mathbf{S} that $\mathbf{S}^2 = \hbar^2 k(k+1)\mathbf{1}$ where k is some integer multiple of $1/2$, and the dimension of \mathcal{K} is $2k+1$. Moreover, two such irreducible angular momenta for the same k are necessarily unitarily equivalent (there is an unitary U such that $\mathcal{K}' = U\mathcal{K}$ and $\mathbf{S}'U = U\mathbf{S}$); this is the reason for the superscript $[k]$. And, finally, for any $k \in \{0, 1/2, 1, 3/2, \dots\}$ there is such an irreducible angular momentum –now called $\mathbf{S}^{[k]-}$ acting on a Hilbert space of dimension $(2k+1)$ conveniently denoted by $\mathfrak{H}^{[k]}$. The story continues as follows: given $\mathbf{S}^{[k]}$ the spectrum of each of its three components is $\{\hbar q : q \in \{-k, -k+1, -k+2, \dots, k\}\}$ so that each eigenvalue is simple (i.e., non-degenerate). There then is, of course, an orthonormal basis $\{|k, q\rangle : q \in \{-k, -k+1, -k+2, \dots, k\}\}$ which diagonalizes the third component $S_3^{[k]}$ but, again due to the angular momentum commutation relations, can also be forced to satisfy $S_{\pm}^{[k]}|k, q\rangle = \hbar\sqrt{k(k+1) - q(q \pm 1)}|k, q \pm 1\rangle$. Such a basis –I will refer to a standard-basis– is unique (!) up to multiplication of all the basis elements by one and the same complex constant of modulus 1; in particular the matrix-elements, i.e. the numbers $D_{q'q}^{(k)}(\alpha, \beta, \gamma)$, are uniquely specified by the single parameter k along with the Euler angles.

Thus, (1) states that an ISTO is a collection of $2k+1$ operators that transform with respect to rotations (generated by some angular momentum namely \mathbf{J}) as do the $2k+1$ elements of a standard basis of \mathbb{C}^{2k+1} upon a ‘rotation’ in this latter space.

Proceeding from (1) the text of Ballentine continues and claims that this equation may be rewritten as (omitting the Euler angles)⁵:

$$(4) \quad \mathbf{R}T_q^{(k)}\mathbf{R}^{-1} = \sum_{q'} T_{q'}^{(k)} \langle k, q' | \mathbf{R} | k, q \rangle ;$$

and this claim is simply wrong because the rotation matrix $D^{(k)}$ is confused with the matrix associated with the rotation (operator) \mathbf{R} ! To show that the right-hand side of (4) is meaningless, consider a pure spin $1/2$, so that in my notation $\mathbf{J} = \mathbf{S}^{[1/2]}$. The operator \mathbf{J} is a vector operator with respect to itself so that transforming into spherical components

$$T_o^{(1)} := J_3, \quad T_{\pm 1}^{(1)} := \mp(J_1 \pm iJ_2)/\sqrt{2},$$

²Frequently and interestingly the components of the angular momentum are unbounded and this necessitates the addition of sufficient salt and pepper in the form of domain questions in order to become mathematically digestible. In a textbook one sticks to the algebraic aspects only which are really the crucial ones.

³Alternatively, Schur's Lemma states that this is equivalent to demanding that the only subspaces of \mathcal{K} which are invariant under all \mathbf{R} 's are the trivial ones \mathcal{K} and 0 .

⁴Due to the angular momentum commutation relations.

⁵The eq. is unnumbered in Ballentine (foot of p.193)

$T_q^{(1)}$ ($q = 1, 0, -1$) is certainly an ISTO of degree 1 with respect to \mathbf{J} . However, the r.h.s of (4) asks us (in order to verify the defining identity) to compute $3 \times 3 = 9$ matrix-elements of an operator acting in a 2-dimensional Hilbert space ! The correct version of the culprit is of course, with (3):

$$(5) \quad \mathbf{R}T_q^{(k)}\mathbf{R}^{-1} = \sum_{q'} T_{q'}^{(k)} \langle k, q' | D^{(k)} | k, q \rangle ;$$

where, to insist again

$$\mathbf{R}(\alpha, \beta, \gamma) = e^{-i\phi\hat{\mathbf{n}}\cdot\mathbf{J}/\hbar} , \quad D^{(k)}(\alpha, \beta, \gamma) = e^{-i\phi\hat{\mathbf{n}}\cdot\mathbf{S}^{[k]}/\hbar} ;$$

\mathbf{R} acts on wherever the reference angular momentum \mathbf{J} is defined whereas $D^{[k]}$ acts in the $(2k+1)$ -dimensional Hilbert space $\mathfrak{H}^{[k]}$ where the irreducible elementary spin of magnitude k is defined.

The error incurred propagates; the infinitesimal version of (4) is then Ballentine's eq. (7.111)

$$(6) \quad [\hat{\mathbf{n}} \cdot \mathbf{J}, T_q^{(k)}] = \sum_{q'} T_{q'}^{(k)} \langle k, q' | \hat{\mathbf{n}} \cdot \mathbf{J} | k, q \rangle$$

where, concomitantly, the r.h.s is meaningless because $\hat{\mathbf{n}} \cdot \mathbf{J}$ acts on the Hilbert space carrying \mathbf{J} and $\{|k, q\rangle\}$ is the same good, old, orthonormal basis of $\mathfrak{H}^{[k]} \equiv \mathbb{C}^{2k+1}$. Of course, differentiating (5) –with (2) and (3) in mind– with respect to the rotation-angle ϕ and setting $\phi = 0$ – we get the correct version, due to Racah,

$$(7) \quad [\hat{\mathbf{n}} \cdot \mathbf{J}, T_q^{(k)}] = \sum_{q'} T_{q'}^{(k)} \langle k, q' | \hat{\mathbf{n}} \cdot \mathbf{S}^{[k]} | k, q \rangle .$$

By an irreproducible tour de force starting from (6), Ballentine lands atop his equations (7.112a,b,c) which is the usual –and perfectly correct infinitesimal version of the definition of an ISTO due to Racah [5]:

$$(8) \quad [J_3, T_q^{(k)}] = \hbar q T_q^{(k)} , \quad [J_{\pm}, T_q^{(k)}] = \hbar \sqrt{k(k+1) - q(q \pm 1)} T_{q \pm 1}^{(k)} .$$

This follows from (7) by choosing $\hat{\mathbf{n}}$ to be first, the unit vector in the z or third direction and then to be the spherical vectors $\hat{\mathbf{n}}_{\pm} := (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})/\sqrt{2}$. It is also obviously equivalent to (7) because these three vectors are just another linearly independent system in \mathbb{C}^3 . Supposedly, the same procedure involving the meaningless formulae

$$\langle k, q' | J_3 | k, q \rangle = \hbar q \delta_{q', q} , \quad \langle k, q' | J_{\pm} | k, q \rangle = \hbar \sqrt{k(k+1) - q(q \pm 1)} \delta_{q', q \pm 1}$$

lead from nonsensical (6) to the correct eq. (8). Notice that the presence and crucial role of the second angular momentum $\mathbf{S}^{[k]}$ is entirely hidden but encoded in the numerical factors on the r.h.s of Racah's equation (8). Ballentine then continues to digress on ISTO's arriving eventually to the Wigner-Eckart Theorem of which a (correct) proof is given. I am not at all prepared to follow through each and every claim or calculation performed after or using the meaningless unnumbered equation (4) and its alter ego (7.111). As far as I was willing to check, the rest of the formulas of 7.8 are correct.

Something entirely similar occurs in Zettili's book. Although the (integrated, global and extremely useful) versions (1) and the false (4) are not presented at all as far as I see, the infamous (6) appears in various guises as eqs. (7.309) all the way up to (7.314) and, then (7.317); all of them nonsensical.

The book by Leslie E. Ballentine's first edition is claimed in the preface to be from 1990 (Prentice-Hall) although the World-Scientific edition is from 1998, with various reprints (the copy I examined is from 2001). The book by Nouredine Zettili is a publication of 2001; in its preface it claims to be, basically, a problem solving book, and Ballentine's book is cited. Is there something special about these two specific books mentioned here? Do both have some common ancestor? Have they been reviewed in the relevant physics journals? Why have these (and other) mistakes not been corrected? What about other "modern" books on quantum mechanics? I do not know and I am not willing to lose much time in finding out. But both are sold for good money on the recommendation of learned professors to young people eager to discover the subtleties of quantum theory.

I certainly do not see any reason whatsoever to replace my Dover edition of Messiah (two volumes bound as one) by anything more modern⁶. And I plea: stop this propagation of nonsense.

References

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- [5] Racah, G.: Phys. Rev. **61**, 186 (1942); Phys. Rev. **62**, 438 (1942); Phys. Rev. **63**, 467 (1942).
- [6] Straumann, N.: Quantenmechanik. Nichtrelativistische Quantentheorie. Springer-Verlag, Berlin 2002.
- [7] Wigner, E.: Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren, Vieweg und Sohn, Braunschweig, 1931. English translation as 'Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra' published by Academic Press, New York, in 1959.
- [8] Zettili, N.: Quantum Mechanics. Concepts and Applications. J. Wiley & Sons, Chichester 2001.

⁶Alongside, I also keep Norbert Straumann's "Quantenmechanik" [6] which I gauge as a both modern and excellent (advanced) textbook and the two volumes by Galindo and Pascual [3] which although not a textbook, is a reliable and wonderful Pandora's box.

Examples of spherical harmonics. Wave function of a particle in an eigenstate of the orbital angular momentum. 9.4 Rotation energy of a diatomic molecule. Quantum mechanics, the universal cosmic language. 14.8.1 Life, intelligence, and thought. 14.8.2 Listening to extraterrestrials. 14.8.3 Quantum mechanics, the universal cosmic language. Index. Lectures on Quantum Mechanics. Jean-Louis Basdevant. Lectures on Quantum Mechanics. Professor Jean-Louis Basdevant Physics Department École Polytechnique 91128 Palaiseau France jean-louis.basdevant@polytechnique.edu. Cover illustration: Schrödinger's cat. Modern Quantum Mechanics J. J. Sakurai Wa Z Y WY _ N N oS Revised Edition Modern Quantum Mechanics Revised Edition J. J. Sakurai Late, University of California, Los Angeles San Fu Tuan, Editor University of Hawaii, Manoa Addison-Wesley Publishing Company Reading, Massachusetts * Menlo Park, California * New York Don Mills, Ontario * Wokingham, England Å Amsterdam + Bonn Sydney + Singapore + Tokyo * Madrid + San Juan Å Milan Å Paris. One has but to read his very lucid textbooks on Invariance Principles and Elementary Particles and Advanced Quantum Mechanics as well as his reviews and summer school lectures to appreciate this. Quantum Mechanics Second Edition Quantum Mechanics Concepts and Applications Second Edition Nouredine Zettili Jacksonville State University, Jacksonville, USA A John Wiley and Sons, Ltd., Publication Copyright © 2009 John Wiley & Sons, Ltd Registered office John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom For details of our global editorial offices, for customer services and for information. about how to apply for permission to reuse the copyright material in this book please see our website at www.wiley.com. 428 7.4.4 Wigner-Eckart Theorem for Spherical Tensor Operators . . . While many fine textbooks on quantum mechanics exist, problem-solving books are far fewer. No-Nonsense Quantum Mechanics is a book about well-established topics, and the main focus is to present these topics in the most student-friendly way. However, there are several sections on topics that other beginner textbooks don't mention, even though they are well-established among experts. And I believe students can get much deeper insights by finding out about them at an earlier stage. I'm a non-native speaker. 19. Quantum Mechanics of the Electromagnetic Field. 20. Bell's Theorem & its Consequences. Appendix (A) - Schur's Lemma. The Formulation of Quantum Mechanics 2.1 Basic Theoretical Concepts 2.2 Conditions on Operators 2.3 General States and Pure States 2.4 Probability Distributions Problems. 42 42 48 50 55 60. Chapter 3. Kinematics and Dynamics 3.1 Transformations of States and Observables 3.2 The Symmetries of Space-Time 3.3 Generators of the Galilei Group 3.4 Identification of Operators with Dynamical Variables 3.5 Composite Systems 3.6 [[Quantizing a Classical System]] 3.7 Equations of Motion 3.8 Symmetries and Conservation Laws Problems.