REU COURSE: GAME THEORY

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1. What this course is about

Mathematical Game Theory is concerned with formalizing decision-making situations in order to find optimal choices; that is, ones that yield the highest expected payoff. Started by John Von Neumann and Oskar Morgenstern, and popularized by John Nash, the field has recently attracted the interest of not only mathematicians and economists but also biologists, social scientists, and engineers.

We will begin by studying the game of Hex, a two-person, zero-sum game played on a symmetric board invented by John Nash, which has beautiful applications to plane topology. We will follow the recent book by Yuval Peres *Game Theory, Alive* for the discussion of some of the basic theorems traditionally associated with Game Theory such as various fixed point theorems, Nash and correlated equilibria, and Von Neumann's Minimax Theorem.

Prerequisites: Calculus, Linear Algebra, and familiarity with the basic ideas in Probability. The more advanced ideas in Probability, starting from conditional probability, will be provided in the course.

2. Undergraduate research vision

Probability is key to understand many of the modern developments in Mathematics. Recently Peres-Schramm-Sheffield-Wilson (2008, JAMS) have discovered an explicit connection between martingales and non linear pdes. This can explained in the frame of mean value properties that hold only locally and that can be interpreted using stochastic games. In this approach Game Theory plays the role of *Nonlinear Probability.* Undergraduate students can understand the basic notions in a discrete setting, where measure theory is not needed, and where simulations are relatively easy to set up.

3. Applications and broader impact

Game Theory is often taught in Economics departments given its immediate application to economic behavior. Increasingly the techniques of Game Theory have been used in Engineering, especially in Signal Processing and in Electrical Networks. Students who get a good understanding of Probability and Game Theory, will be very prepared to attend graduate school in a variety of subjects ranging from Economics to Biostatistics.

4. Suggested projects

(1) Analysis of the Factory Game: Suppose three factories are located by a lake. Each year, they have a choice of either polluting it or purifying it. If they choose to pollute the lake, there is no cost to the factories. However, if two or more factories pollute the lake, the water becomes unusable, and each factory has to pay b units to obtain water from another source. To purify the lake, each factory must pay a units. If two or more factories choose to purify the lake, then the water is still usable, and there is no additional cost to the factories (you may assume $b > a > 0$). Consider extensions to n factories.

(2) Tug-of-War on Graphs, discrete ∞-Laplace operator. In the classical tug-of-war games players take turns to move one step in the direction of their choice. Suppose now that the order of move is determined by an unbiased coin; that is, when the coin toss is head one player moves, and when it is tails the other player moves. This stochastic tug-of-war game turns out to be surprisingly more regular than anticipated. If we assigned a pay-off function to the boundary vertices of the graph, the value function of the stochastic tug-of-war game turns out to be the solution to the Dirichlet problem for the discrete ∞-Laplace operator on the graph with those boundary values! Compare this to the same phenomena when you just move at random. In this case you would be solving the Dirichlet problem for the regular Laplace operator.

We will focus on the case of directed trees, where there are explicit formulas for the solution of the Dirichlet problem both for the regular Laplacian and for the ∞-Laplacian.

(3) Escaping to the boundary. First start with random walks in one dimension and establish that unless then the random walk is unbiased, we always go to the side with the larger probability. Then consider random walks plus tug-of-war in one and several dimensions. The answer is then not clear, depending on the relative size of various parameters, This project will first run simulations to get ideas on what we can attempt to prove rigorously.

5. The tools of the trade

(1) *Internet tools:* Use Google and Wikipedia, but don't believe anything you get from the web without double checking and crossreferencing.

(2) *LaTeX* is the de-facto standard for mathematical typesetting. It is available free for all major computer platforms.
MikTex is the most popular implementation for Windows:
`http://miktex.org/`
MacTex is the most popular implementation for Mac OS:

`http://www.tug.org/mactex/`

For Ubuntu and other Linux systems:

`https://help.ubuntu.com/community/LaTeX`

(3) *Mathematica:* One of several computer environment to do mathematics by computer. It is an expensive professional program which will be available to REU participants.
(4) *SAGE:* This is the future of doing mathematics by computer. It is a free open source alternative to the commercial programs *Mathematica, Maple, and Matlab.*

`http://www.sagemath.org/`

6. References

[L] Lawler, Gregory F., *Topics in Probability and Analysis*, Lectures from the University of Chicago REU Program, 2009, `http://www.math.uchicago.edu/ lawler/reu.pdf`

[MOS] J.J. Manfredi, A. Oberman, and A. Sviridov, *Nonlinear Elliptic Partial Differential Equations and p-harmonious Functions on Graphs.* Preprint 2012.

[MS2] A. P. Maitra, and W. D. Sudderth, *Discrete gambling and stochastic games.* Applications of Mathematics 32, Springer, New York, 1996.

[MPR] J. J. Manfredi, M. Parviainen and J. D. Rossi, *An asymptotic mean value characterization of p-harmonic functions.* Proc. Amer. Math. Soc., 138:881–889, 2010.

[MPR2] J. J. Manfredi, M. Parviainen and J. D. Rossi, *On the definition and properties of p-harmonious functions.* Preprint.

[MPR3] J. J. Manfredi, M. Parviainen and J. D. Rossi, *Dynamic programming principle for tug-of-war games with noise.* Preprint

[O] A. M. Oberman, *A convergent difference scheme for the infinity Laplacian: construction of absolutely minimizing Lipschitz extensions.* Math. Comp. 74(251):1217–1230, 2005.

[PSSW] Y. Peres, O. Schramm, S. Sheffield and D. Wilson; *Tug-of-war and the infinity Laplacian.* J. Amer. Math. Soc., 22:167–210, 2009.

[PS] Y. Peres, S. Sheffield; *Tug-of-war with noise: a game theoretic view of the p-Laplacian.* Duke Math. J., 145(1):91–120, 2008.

Game Theory. SOE-YCS0002. Stanford School of Engineering. Description. Popularized by movies such as "A Beautiful Mind," game theory is the mathematical modeling of strategic interaction among rational (and irrational) agents.Â The course will provide the basics: representing games and strategies, the extensive form (which computer scientists call game trees), Bayesian games (modeling things like auctions), repeated and stochastic games, and more. We'll include a variety of examples including classic games and a few applications. You can find a full syllabus and description of the course here: http://web.stanford.edu/~jacksonm/GTOC-Syllabus.html. You can find an introductory video here: http://web.stanford.edu/~jacksonm/Intro_Networks.mp4. Instructors. Game Theory. Category 'Best Course for Broadening Horizons and Diversity of Knowledge and Skills'. Type: Elective course (Management). Area of studies: Management. Delivered by: School of Economics and Finance.Â Have skills in the analysis of economic and managerial phenomena and processes using game-theoretic models. Know the basic concepts and theorems of game theory, know algorithms and methods applied to solve business and managerial problems. Understand limits and conditions for applying each game solution concept. Have skills in the analysis of economic and managerial phenomena and processes using game-theoretic models.Â Basic elements of game in a normal form, games classification with examples. Prisoners dilemma: idea, analysis with 4 players with different preferences, basic rules of game theory. 2. Strict and weak dominance. Definition of strict and weak dominance, strict and weak dominated and dominating strategies. Introduction, overview, uses of game theory, some applications and examples, and formal definitions of: the normal form, payoffs, strategiesÂ Popularized by movies such as "A Beautiful Mind," game theory is the mathematical modeling of strategic interaction among rational (and irrational) agents. Beyond what we call `games' in common language, such as chess, poker, soccer, etc., it includes the modeling of conflict among nations, political campaigns, competition among firms, and trading behavior in markets such as the NYSE. How could you begin to model keyword auctions, and peer to peer file-sharing networks, without accounting for the incentives of the people using them?