Geometric Constructions

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Dedicated to Professor M.K. Siu

1. Introduction

A good picture is worth more than a thousand words. This is especially true for students and teachers of geometry. With good illustrations, concepts and problems in geometry are more transparent and understandable. The difficulty of drawing good blackboard geometric sketches is nevertheless well appreciated by every teacher of mathematics. At the same time, for many fascinating problems of constructions using ruler and compass, it is often impractical to carry out detailed constructions with paper and pencil, so much so that in many cases one is forced to settle for constructibility. For example, Howard Eves, in his solution [6] of the euclidean construction of a triangle given the lengths of a side and the median and angle bisector on the same side, made the following remark after proving constructibility.

> The devotee of the game of Euclidean constructions is not really interested in the actual mechanical construction of the sought triangle, but merely in the assurance that the construction is possible. To use a phrase of Jacob Steiner, the devotee performs his construction "simply by means of the tongue" rather than with actual instruments on paper.

The availability in recent years of computer software on dynamic geometry has brought about a change of attitude. Beautiful and accurate geometric diagrams can now be drawn, edited, and organized efficiently on computer screens. This new technological capability stimulates the desire to strive for elegance in actual geometric constructions. [1] Pedagogically, such analysis and explicit constructions provide a fruitful alternative to the traditional emphasis of the deductive method in the learning and teaching of geometry.

In this paper we present a fantasia of euclidean constructions whose analysis make use of elementary algebra and very basic knowledge of euclidean geometry. We focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad®. The tremendous improvement on the economy of time and effort is hard to exaggerate. The most remarkable feature of the Geometer's Sketchpad® is the capability of customizing a tool folder to make constructions as efficient as one would like. Common, basic constructions need only be performed once, and saved as tools for future use. We shall use the Geometer's Sketchpad® simply as ruler and compass, assuming a tool folder

[1] See §7.1 for an explicit construction of the triangle above with a given side, median, and angle bisector.

containing at least the following tools[2] for ready use:
(i) basic shapes such as equilateral triangle and square,
(ii) tangents to a circle from a given point,
(iii) circumcircle and incircle of a triangle.

2. Some examples

We present a few examples of constructions whose elegance is suggested by an analysis a little more detailed than is necessary for constructibility or routine constructions. A number of constructions in this paper are based on diagrams in the interesting book [8]. We adopt the following notation for circles:
(i) $A(r)$ denotes the circle with center A, radius r;
(ii) $A(B)$ denotes the circle with center A, passing through the point B, and
(iii) (A) denotes a circle with center A and unspecified radius, but unambiguous in context.

2.1. Construct a regular octagon by cutting corners from a square.

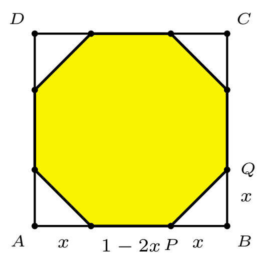

Figure 1A

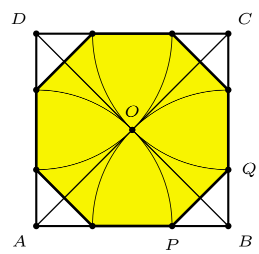

Figure 1B

Suppose an isosceles right triangle of (shorter) side x is to be cut from each corner of a unit square to make a regular octagon. See Figure 1A. A simple calculation shows that $x = 1 - \frac{\sqrt{2}}{2}$. This means $AP = 1 - x = \frac{\sqrt{2}}{2}$. The point P, and the other vertices, can be easily constructed by intersecting the sides of the square with quadrants of circles with centers at the vertices of the square and passing through the center O. See Figure 1B.

2.2. The centers A and B of two circles lie on the other circle. Construct a circle tangent to the line AB, to the circle (A) internally, and to the circle (B) externally.

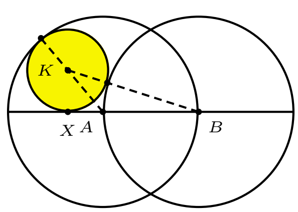

Figure 2A

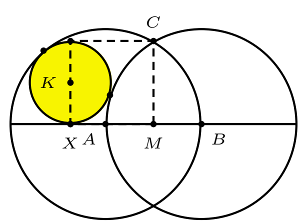

Figure 2B

[2]A construction appearing in sans serif is assumed to be one readily performable with a customized tool.

Suppose $AB = a$. Let $r =$ radius of the required circle (K), and $x = AX$, where X is the projection of the center K on the line AB. We have

$$(a+r)^2 = r^2 + (a+x)^2, \tag{1}$$

$$(a-r)^2 = r^2 + x^2. \tag{2}$$

Subtraction gives $4ar = a^2 + 2ax$ or $x + \frac{a}{2} = 2r$. Putting $x = 2r - \frac{a}{2}$ into (2) we easily find that $r = \frac{\sqrt{3}}{4}a$. Therefore $x + \frac{a}{2} = 2r = \frac{\sqrt{3}}{2}a$. This means that the diameter through K is a side of the square on CM, and the circle can be easily constructed. See Figure 2B.

2.3. *Equilateral triangle in a rectangle.* Given a rectangle $ABCD$, construct points P and Q on BC and CD respectively such that triangle APQ is equilateral.

Construction 1. *Construct* equilateral triangles CDX *and* BCY*, with* X *and* Y *inside the rectangle. Extend* AX *to intersect* BC *at* P *and* AY *to intersect* CD *at* Q*.*

The triangle APQ *is equilateral.* See Figure 3B.

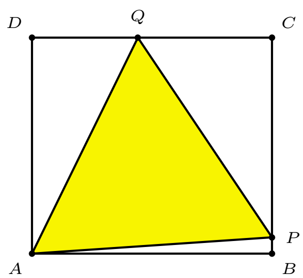

Figure 3A

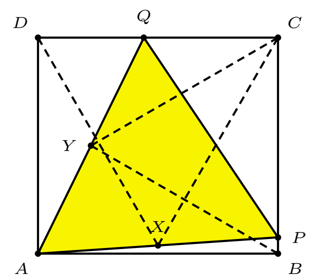

Figure 3B

This construction did not come from a lucky insight. It was found by an analysis. Let $AB = DC = a$, $BC = AD = b$. If $BP = y$, $DQ = x$ and APQ is equilateral, then a calculation shows that $x = 2a - \sqrt{3}b$ and $y = 2b - \sqrt{3}a$. From these expressions of x and y the above construction was devised.

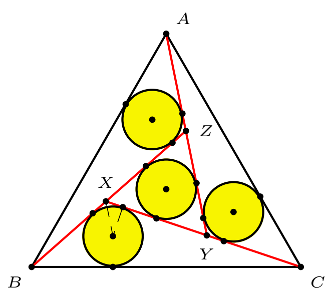

Figure 4A

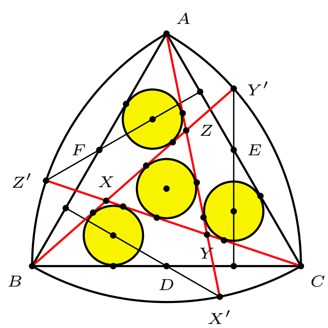

Figure 4B

2.4. *Partition of an equilateral triangle into 4 triangles with congruent incircles.* Given an equilateral triangle, construct three lines each through a vertex so that the incircles of the four triangles formed are congruent. See Figure 4A and [8, Problem 2.1.7] and [9, Problem

5.1.3], where it is shown that if each side of the equilateral triangle has length a, then the small circles all have radii $\frac{1}{8}(\sqrt{7}-\sqrt{3})a$. Here is a calculation that leads to a very easy construction of these lines.

In Figure 4A, let $CX = AY = BZ = a$ and $BX = CY = AZ = b$. The equilateral triangle XYZ has sidelength $a-b$ and inradius $\frac{\sqrt{3}}{6}(a-b)$. Since $\angle BXC = 120^\circ$, $BC = \sqrt{a^2+ab+b^2}$, and the inradius of triangle BXC is

$$\frac{1}{2}(a+b-\sqrt{a^2+ab+b^2})\tan 60^\circ = \frac{\sqrt{3}}{2}(a+b-\sqrt{a^2+ab+b^2}).$$

These two inradii are equal if and only if $3\sqrt{a^2+ab+b^2} = 2(a+2b)$. Applying the law of cosines to triangle XBC, we obtain

$$\cos XBC = \frac{(a^2+ab+b^2)+b^2-a^2}{2b\sqrt{a^2+ab+b^2}} = \frac{a+2b}{2\sqrt{a^2+ab+b^2}} = \frac{3}{4}.$$

In Figure 4B, Y' is the intersection of the arc $B(C)$ and the perpendicular from the midpoint E of CA to BC. The line BY' makes an angle $\arccos\frac{3}{4}$ with BC. The other two lines AX' and CZ' are similarly constructed. These lines bound the equilateral triangle XYZ, and the four incircles can be easily constructed. Their centers are simply the reflections of X' in D, Y' in E, and Z' in F.

3. Some basic constructions

3.1. *Geometric mean and the solution of quadratic equations.* The following constructions of the geometric mean of two lengths are well known.

Construction 2. (a) *Given two segments of length a, b, mark three points A, P, B on a line (P between A and B) such that $PA = a$ and $PB = b$. Describe a semicircle with AB as diameter, and let the perpendicular through P intersect the semicircle at Q. Then $PQ^2 = AP \cdot PB$, so that the length of PQ is the geometric mean of a and b.* See Figure 5A.

(b) *Given two segments of length $a < b$, mark three points P, A, B on a line such that $PA = a$, $PB = b$, and A, B are on the* same *side of P. Describe a semicircle with PB as diameter, and let the perpendicular through A intersect the semicircle at Q. Then $PQ^2 = PA \cdot PB$, so that the length of PQ is the geometric mean of a and b.* See Figure 5B.

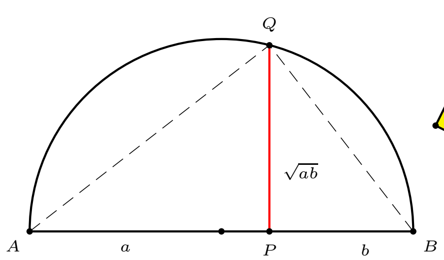

Figure 5A

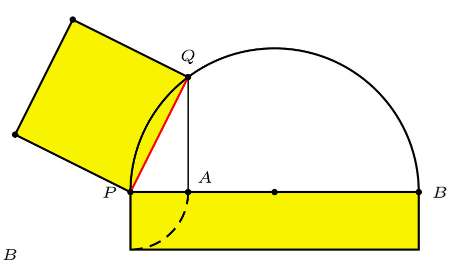

Figure 5B

More generally, a quadratic equation can be solved by applying the theorem of intersecting chords: *If a line through P intersects a circle $O(r)$ at X and Y, then the product $XP \cdot PY$ (of signed lengths) is equal to $r^2 - OP^2$.* Thus, if two chords AB and XY

intersect at P, then $AP \cdot PB = XP \cdot PY$. See Figure 6A. In particular, if P is outside the circle, and if PT is a tangent to the circle, then $PT^2 = PX \cdot PY$ for any line intersecting the circle at X and Y. See Figure 6B.

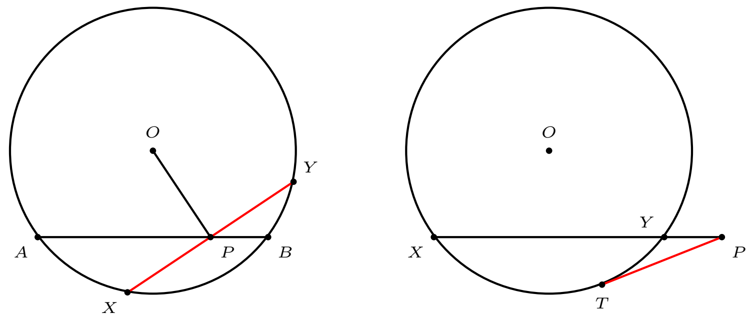

Figure 6A Figure 6B

The algebraic method of the solution of a quadratic equation by completing squares can be easily incorporated geometrically by using the Pythagorean theorem. We present an example.

3.1.1. Given a chord BC perpendicular to a diameter XY of circle (O), to construct a line through X which intersects the circle at A and BC at T such that AT has a given length t. Clearly, $t \leq YM$, where M is the midpoint of BC.

Let $AX = x$. Since $\angle CAX = \angle CYX = \angle TCX$, the line CX is tangent to the circle ACT. It follows from the theorem of intersecting chords that $x(x-t) = CX^2$. The method of completing squares leads to

$$x = \frac{t}{2} + \sqrt{CX^2 + \left(\frac{t}{2}\right)^2}.$$

This suggests the following construction.[3]

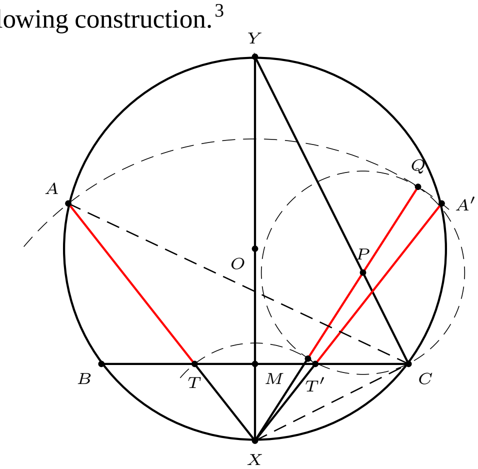

Figure 7

[3] This also solves the construction problem of triangle ABC with given angle A, the lengths a of its opposite side, and of the bisector of angle A.

Construction 3. *On the segment CY, choose a point P such that $CP = \frac{t}{2}$. Extend XP to Q such that $PQ = PC$. Let A be an intersection of $X(Q)$ and (O). If the line XA intersects BC at T, then $AT = t$.* See Figure 7.

3.2. *Harmonic mean and the equation $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$.* The harmonic mean of two quantities a and b is $\frac{2ab}{a+b}$. In a trapezoid of parallel sides a and b, the parallel through the intersection of the diagonals intercepts a segment whose length is the harmonic mean of a and b. See Figure 8A. We shall write this harmonic mean as $2t$, so that $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$. See Figure 8B.

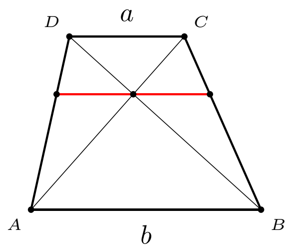

Figure 8A

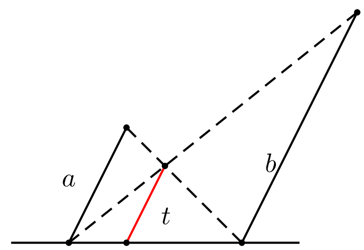

Figure 8B

Here is another construction of t, making use of the formula for the length of an angle bisector in a triangle. If $BC = a$, $AC = b$, then the angle bisector CZ has length

$$t_c = \frac{2ab}{a+b} \cos \frac{C}{2} = 2t \cos \frac{A}{2}.$$

The length t can therefore be constructed by completing the rhombus $CXZY$ (by constructing the perpendicular bisector of CZ to intersect BC at X and AC at Y). See Figure 9A. In particular, if the triangle contains a right angle, this trapezoid is a square. See Figure 9B.

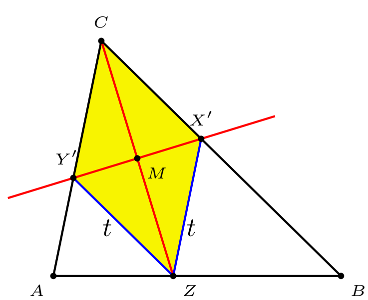

Figure 9A

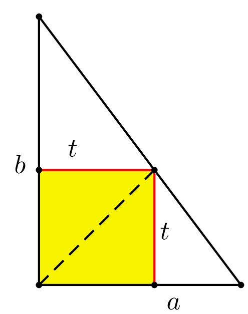

Figure 9B

4. The shoemaker's knife

4.1. *Archimedes' Theorem.* A shoemaker's knife (or arbelos) is the region obtained by cutting out from a semicircle with diameter AB the two smaller semicircles with diameters

AP and PB. Let $AP = 2a$, $PB = 2b$, and the common tangent of the smaller semicircles intersect the large semicircle at Q. The following remarkable theorem is due to Archimedes. See [11].

Theorem 1 (Archimedes)**.** (1) *The two circles each tangent to PQ, the large semicircle and one of the smaller semicircles have equal radii $t = \frac{ab}{a+b}$.* See Figure 10A.

(2) *The circle tangent to each of the three semicircles has radius*

$$\rho = \frac{ab(a+b)}{a^2 + ab + b^2}. \tag{3}$$

See Figure 10B.

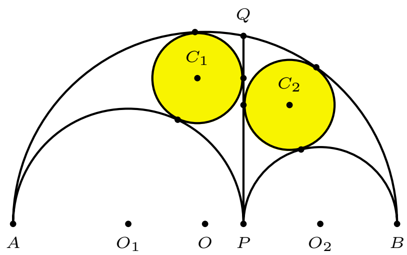

Figure 10A

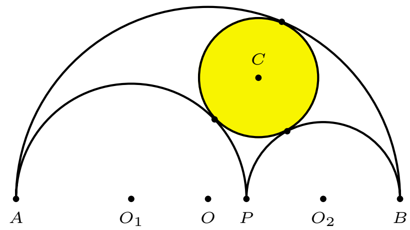

Figure 10B

Here is a simple construction of the Archimedean "twin circles". Let Q_1 and Q_2 be the "highest" points of the semicircles $O_1(a)$ and $O_2(b)$ respectively. The intersection of $C_3 = O_1Q_2 \cap O_2Q_1$ is a point "above" P, and $C_3P = t = \frac{ab}{a+b}$.

Construction 4. *Construct the circle $P(C_3)$ to intersect the diameter AB at P_1 and P_2* (*so that P_1 is on AP and P_2 is on PB*).

The center C_1 (*respectively C_2*) *is the intersection of the circle $O_1(P_2)$* (*respectively $O_2(P_1)$*) *and the perpendicular to AB at P_1* (*respectively P_2*). See Figure 11.

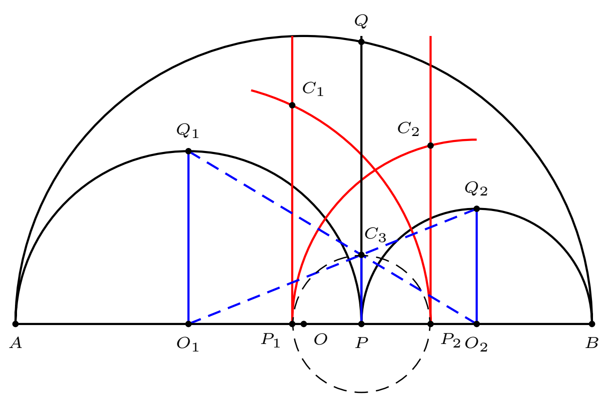

Figure 11

Theorem 2 (Bankoff [3])**.** *If the incircle $C(\rho)$ of the shoemaker's knife touches the smaller semicircles at X and Y, then the circle through the points P, X, Y has the same radius t as the Archimedean circles.* See Figure 12.

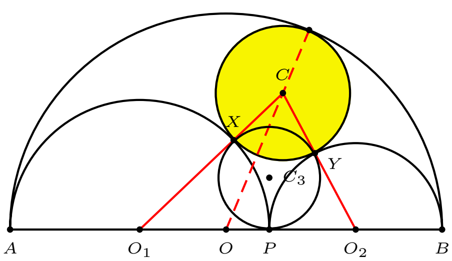

Figure 12

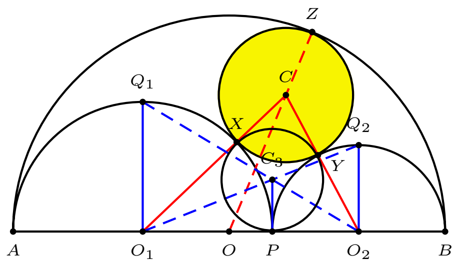

Figure 13

This gives a very simple construction of the incircle of the shoemaker's knife. See Figure 13.

Construction 5. *Let $X = C_3(P) \cap O_1(a)$, $Y = C_3(P) \cap O_2(b)$, and $C = O_1X \cap O_2Y$. The circle $C(X)$ is the incircle of the shoemaker's knife. It touches the large semicircle at $Z = OC \cap O(a+b)$.*

A rearrangement of (3) in the form

$$\frac{1}{a+b} + \frac{1}{\rho} = \frac{1}{t}$$

leads to another construction of the incircle (C) by directly locating the center and one point on the circle. See Figure 14.

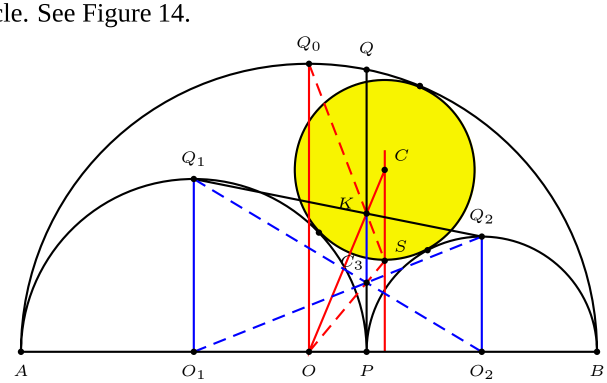

Figure 14

Construction 6. *Let Q_0 be the "highest" point of the semicircle $O(a+b)$. Construct*
(i) $K = Q_1Q_2 \cap PQ$,
(ii) $S = OC_3 \cap Q_0K$, *and*
(iii) *the perpendicular from S to AB to intersect the line OK at C.*
The circle $C(S)$ is the incircle of the shoemaker's knife.

4.2. *Other simple constructions of the incircle of the shoemaker's knife.* We give four more simple constructions of the incircle of the shoemaker's knife. The first is by Leon Bankoff [1]. The remaining three are by Peter Woo [16].

Construction 7 (Bankoff)**.** (1) *Construct the circle $Q_1(A)$ to intersect the semicircles $O_2(b)$ and $O(a+b)$ at X and Z respectively.*

(2) *Construct the circle* $Q_2(B)$ *to intersect the semicircles* $O_1(a)$ *and* $O(a+b)$ *at* Y *and the same point* Z *in* (1) *above.*

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 15.

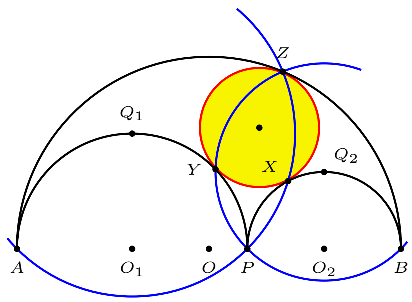

Figure 15

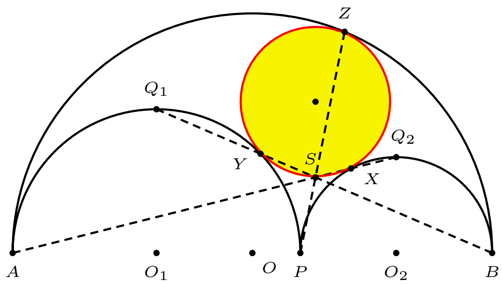

Figure 16

Construction 8 (Woo)**.** (1) *Construct the line* AQ_2 *to intersect the semicircle* $O_2(b)$ *at* X.

(2) *Construct the line* BQ_1 *to intersect the semicircle* $O_1(a)$ *at* Y.

(3) *Let* $S = AQ_2 \cap BQ_1$. *Construct the line* PS *to intersect the semicircle* $O(a+b)$ *at* Z.

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 16.

Construction 9 (Woo)**.** *Let* M *be the "lowest" point of the circle* $O(a+b)$. *Construct*
(i) *the circle* $M(A)$ *to intersect* $O_1(a)$ *at* Y *and* $O_2(b)$ *at* X,
(ii) *the line* MP *to intersect the semicircle* $O(a+b)$ *at* Z.

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 17.

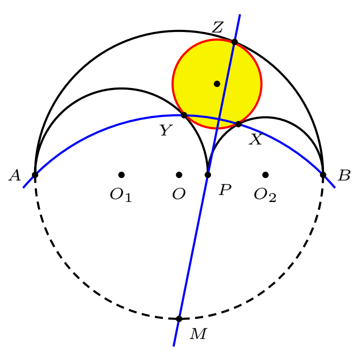

Figure 17

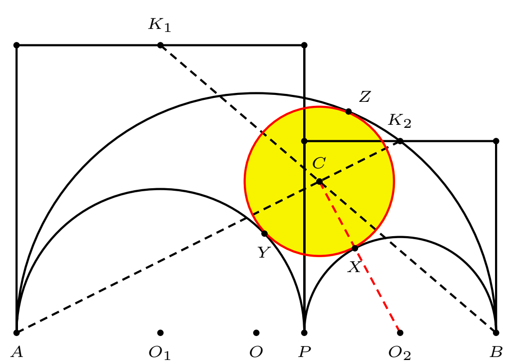

Figure 18

Construction 10 (Woo)**.** *Construct* squares *on* AP *and* PB *on the same side of the shoemaker knife. Let* K_1 *and* K_2 *be the midpoints of the opposite sides of* AP *and* PB *respectively. Let* $C = AK_2 \cap BK_1$, *and* $X = CO_2 \cap O_2(b)$.

The circle $C(X)$ *is the incircle of the shoemaker's knife.* See Figure 18.

5. Animation of bicentric polygons

A famous theorem of J. V. Poncelet states that if between two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ there is a polygon of n sides with vertices on $\mathcal{C}_1$ and sides tangent to $\mathcal{C}_2$, then there is one such polygon of n sides with a vertex at an arbitrary point on $\mathcal{C}_1$. See, for example, [5]. For circles $\mathcal{C}_1$ and $\mathcal{C}_2$ and for $n = 3, 4$, we illustrate this theorem by constructing animation pictures based on simple metrical relations.

5.1. *Euler's formula.* Consider the construction of a triangle given its circumcenter O, incenter I and a vertex A. The circumcircle is $O(A)$. If the line AI intersects this circle again at X, then the vertices B and C are simply the intersections of the circles $X(I)$ and $O(A)$. See Figure 19A. This leads to the famous Euler formula

$$d^2 = R^2 - 2Rr, \tag{4}$$

where d is the distance between the circumcenter and the incenter.

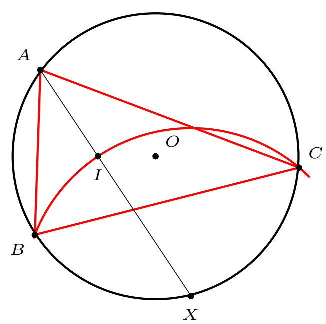

Figure 19A

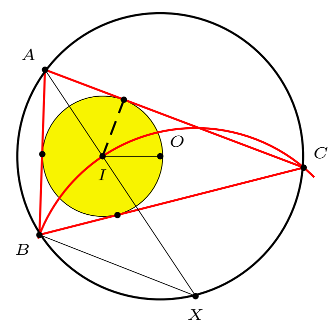

Figure 19B

Proof. If I is the incenter, then $AI = \frac{r}{\sin\frac{A}{2}}$ and $IX = IB = \frac{2R}{\sin\frac{A}{2}}$. See Figure 19B. The power of I with respect to the circumcircle is

$$R^2 - d^2 = AI \cdot IX = r \sin\frac{A}{2} \cdot \frac{2R}{\sin\frac{A}{2}} = 2Rr.$$

□

5.1.1. Given a circle $O(R)$ and $r < \frac{R}{2}$, to construct a point I such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a triangle.

Construction 11. *Let $P(r)$ be a circle tangent to (O) internally. Construct a line through O* tangent to the circle $P(r)$ *at a point I.*

The circle $I(r)$ is the incircle of triangles which have $O(R)$ as circumcircle. See Figure 20.

5.1.2. Given a circle $O(R)$ and a point I, to construct a circle $I(r)$ such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a triangle.

Construction 12. *Construct the circle $I(R)$ to intersect $O(R)$ at a point P, and construct the line PI to intersect $O(R)$ again at Q. Let T be the midpoint of IQ.*

The circle $I(T)$ is the incircle of triangles which have $O(R)$ as circumcircle. See Figure 21.

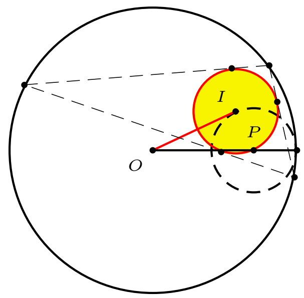

Figure 20

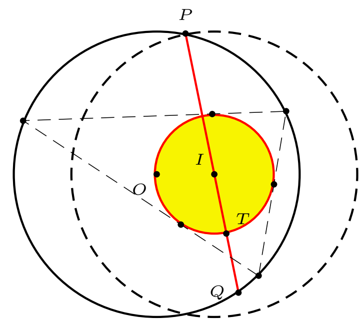

Figure 21

5.1.3. Given a circle $I(r)$ and a point O, to construct a circle $O(R)$ which is the circumcircle of triangles with $I(r)$ as incircle. Since

$$R = r + \sqrt{r^2 + d^2}$$

by the Euler formula (4), we have the following construction. See Figure 22.

Construction 13. *Let IP be a radius of $I(r)$ perpendicular to IO. Extend OP to a point A such that $PA = r$.*

The circle $O(A)$ is the circumcircle of triangles which have $I(r)$ as incircle.

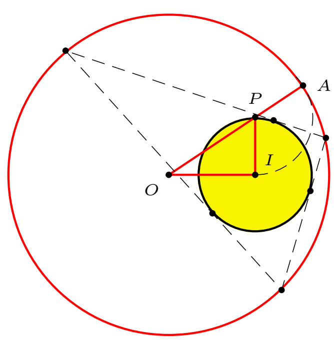

Figure 22

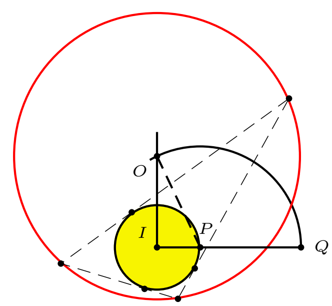

Figure 23

5.1.4. Given $I(r)$ and $R > 2r$, to construct a point O such that $O(R)$ is the circumcircle of triangles with $I(r)$ as incircle.

Construction 14. *Extend a radius IP to Q such that $IQ = R$. Construct the perpendicular to IP at I to intersect the circle $P(Q)$ at O.*

The circle $O(R)$ is the circumcircle of triangles which have $I(r)$ as incircle. See Figure 23.

5.2. *Bicentric quadrilaterals.* A bicentric quadrilateral is one which admits a circumcircle and an incircle. The construction of bicentric quadrilaterals is based on the Fuss formula

$$2r^2(R^2+d^2) = (R^2-d^2)^2, \tag{5}$$

where d is the distance between the circumcenter and incenter of the quadrilateral. See [7, §39].

5.2.1. Given a circle $O(R)$ and a point I, to construct a circle $I(r)$ such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a quadrilateral.

The Fuss formula (5) can be rewritten as

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}.$$

In this form it admits a very simple interpretation: r can be taken as the altitude on the hypotenuse of a right triangle whose shorter sides have lengths $R \pm d$. See Figure 24.

Construction 15. *Extend IO to intersect $O(R)$ at a point A. On the perpendicular to IA at I construct a point K such that $IK = R - d$. Construct the altitude IP of the right triange AIK.*

The circles $O(R)$ and $I(P)$ are the circumcircle and incircle of bicentric quadrilaterals.

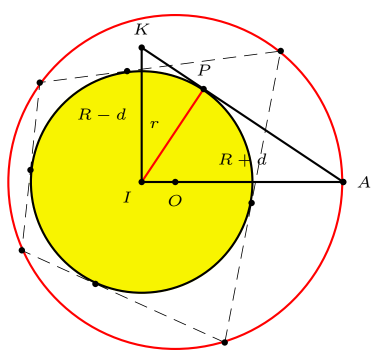

Figure 24

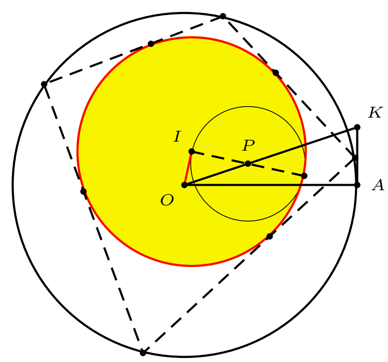

Figure 25

5.2.2. Given a circle $O(R)$ and a radius $r \leq \frac{R}{\sqrt{2}}$, to construct a point I such that $I(r)$ is the incircle of quadrilaterals inscribed in $O(R)$, we rewrite the Fuss formula (5) in the form

$$d^2 = \left(\sqrt{R^2+\frac{r^2}{4}} - \frac{r}{2}\right)\left(\sqrt{R^2+\frac{r^2}{4}} - \frac{3r}{2}\right).$$

This leads to the following construction. See Figure 25.

Construction 16. *Construct a right triangle OAK with a right angle at A, $OA = R$ and $AK = \frac{r}{2}$. On the hypotenuse OK choose a point P such that $KP = r$. Construct a* tangent from O to the circle $P(\frac{r}{2})$. *Let I be the point of tangency.*

The circles $O(R)$ and $I(r)$ are the circumcircle and incircle of bicentric quadrilaterals.

5.2.3. Given a circle $I(r)$ and a point O, to construct a circle (O) such that these two circles are respectively the incircle and circumcircle of a quadrilateral. Again, from the Fuss formula (5),

$$R^2 = \left(\sqrt{d^2 + \frac{r^2}{4}} + \frac{r}{2}\right)\left(\sqrt{d^2 + \frac{r^2}{4}} + \frac{3r}{2}\right).$$

Construction 17. *Let E be the midpoint of a radius IB perpendicular to OI. Extend the ray OE to a point F such that $EF = r$. Construct a* tangent OT to the circle $F\left(\frac{r}{2}\right)$. *Then OT is a circumradius.*

6. Some circle constructions

6.1. *Circles tangent to a chord at a given point.* Given a point P on a chord BC of a circle (O), there are two circles tangent to BC at P, and to (O) internally. The radii of these two circles are $\frac{BP\cdot PC}{2(R\pm h)}$, where h is the distance from O to BC. They can be constructed as follows.

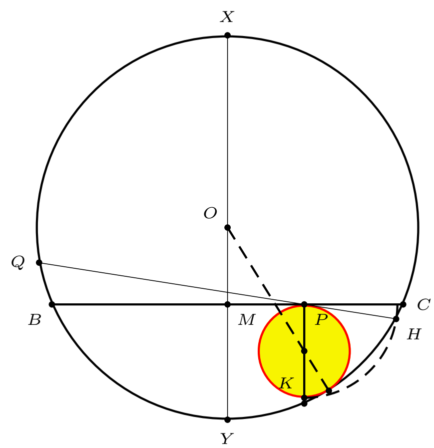

Figure 26A

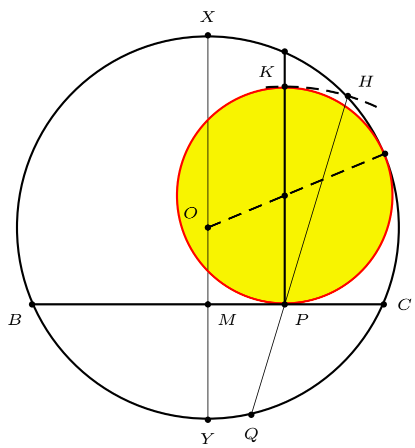

Figure 26B

Construction 18. *Let M be the midpoint of BC, and XY be the diameter perpendicular to BC. Construct*
(i) *the circle center P, radius MX to intersect the arc BXC at a point Q,*
(ii) *the line PQ to intersect the circle (O) at a point H,*
(iii) *the circle $P(H)$ to intersect the line perpendicular to BC at P at K* (*so that H and K are on the same side of BC*).

The circle with diameter PK is tangent to the circle (O). See Figure 26A.

Replacing X by Y in (i) *above we obtain the other circle tangent to BC at P and internally to (O).* See Figure 26B.

6.2. *Chain of circles tangent to a chord.* Given a circle (Q) tangent internally to a circle (O) and to a chord BC at a given point P, there are two neighbouring circles tangent to (O) and to the same chord. These can be constructed easily by observing that in Figure 27, the common tangent of the two circles cut out a segment whose midpoint is B. If (Q') is a neighbour of (Q), their common tangent passes through the midpoint M of the arc BC complementary to (Q). See Figure 28.

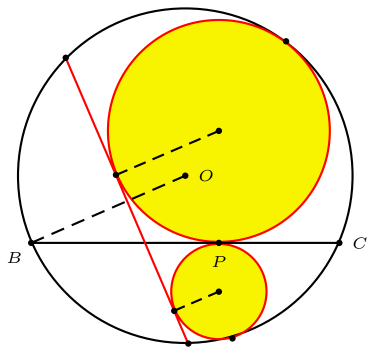

Figure 27

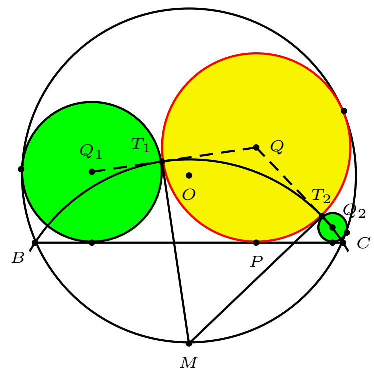

Figure 28

Construction 19. *Given a circle (Q) tangent to (O) and to the chord BC, construct*
(i) *the circle $M(B)$ to intersect (Q) at T_1 and T_2, MT_1 and MT_2 being tangents to (Q),*
(ii) *the bisector of the angle between MT_1 and BC to intersect the line QT_1 at Q_1.*
The circle $Q_1(T_1)$ is tangent to (O) and to BC.
Replacing T_1 by T_2 in (ii) *we obtain Q_2. The circle $Q_2(T_2)$ is also tangent to (O) and BC.*

6.3. *Mixtilinear incircles.* Given a triangle ABC, we construct the circle tangent to the sides AB, AC, and also to the circumcircle internally. Leon Bankoff [4] called this the A- mixtilinear incircle of the triangle. Its center is clearly on the bisector of angle A. Its radius is $r\sec^2\frac{A}{2}$, where r is the inradius of the triangle. The mixtilinear incircle can be constructed as follows. See Figure 29.

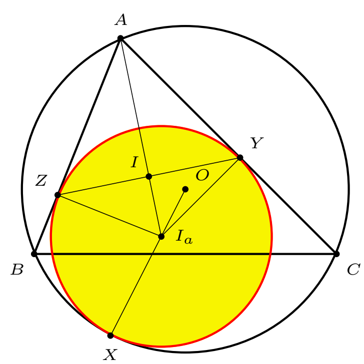

Figure 29

Construction 20 (Mixtilinear incircle)**.** *Let I be the* incenter *of triangle ABC. Construct*
(i) *the perpendicular to IA at I to intersect AC at Y,*
(ii) *the perpendicular to AY at Y to intersect the line AI at I_a.*
The circle $I_a(Y)$ is the A-mixtilinear incircle of ABC.

The other two mixtilinear incircles can be constructed in a similar way. For another construction, see [17].

6.4. *Ajima's construction.* The interesting book [9] by Fukagawa and Rigby contains a very useful formula which helps perform easily many constructions of inscribed circles which are otherwise quite difficult.

Theorem 3 (Ajima)**.** *Given triangles ABC with circumcircle (O) and a point P such that A and P are on the same side of BC, the circle tangent to the lines PB, PC, and to the circle (O) internally is the image of the incircle of triangle PBC under the homothety with center P and ratio $1 + \tan\frac{A}{2}\tan\frac{BPC}{2}$.*

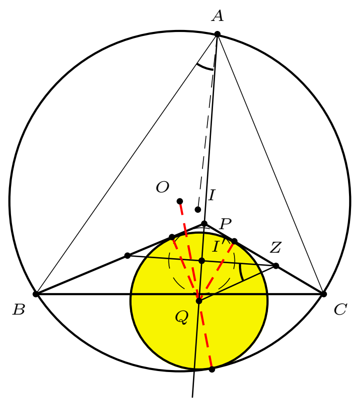

Figure 30

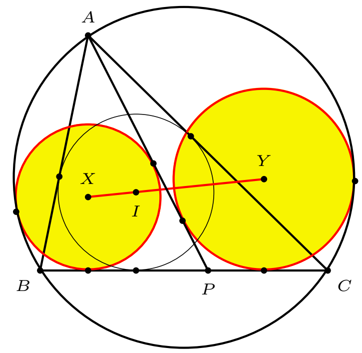

Figure 31

Construction 21 (Ajima)**.** *Given two points B and C on a circle (O) and an arbitrary point P, construct*
(i) *a point A on (O) on the same side of BC as P*, (*for example, by taking the midpoint M of BC, and intersecting the ray MP with the circle (O)*),
(ii) *the* incenter *I of triangle ABC,*
(iii) *the* incenter *I' of triangle PBC,*
(iv) *the perpendicular to $I'P$ at I' to intersect PC at Z.*
(v) *Rotate the ray ZI' about Z through an* (*oriented*) *angle equal to angle BAI to intersect the line AP at Q.*
Then the circle with center Q, tangent to the lines PB and PC, is also tangent to (O) internally. See Figure 30.

6.4.1. *Thébault's theorem.* With Ajima's construction, we can easily illustrate the famous Thébault theorem. See [13, 2] and Figure 31.

Theorem 4 (Thébault)**.** *Let P be a point on the side BC of triangle ABC. If the circles (X) and (Y) are tangent to AP, BC, and also internally to the circumcirle of the triangle, then the line XY passes through the incenter of the triangle.*

6.4.2. *Another example.* We construct an animation picture based on Figure 32 below. Given a segment AB and a point P, construct the squares $APX'X$ and $BPY'Y$ on the segments AP and BP. The locus of P for which A, B, X, Y are concyclic is the union of the perpendicular bisector of AB and the two quadrants of circles with A and B as endpoints. Consider P on one of these quadrants. The center of the circle $ABYX$ is the center of the other quadrant. Applying Ajima's construction to the triangle XAB and the point P, we easily obtain the circle tangent to AP, BP, and (O). Since $\angle APB = 135^\circ$ and $\angle AXB = 45^\circ$, the radius of this circle is twice the inradius of triangle APB. See Figure 32.

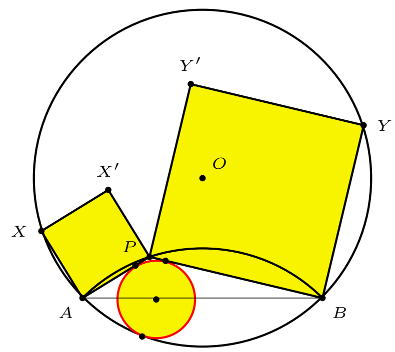

Figure 32

7. Some examples of triangle constructions

There is an extensive literature on construction problems of triangles with certain given elements such as angles, lengths, or specified points. Wernick [14] outlines a project of such with three given specific points. Lopes [12], on the other hands, treats extensively the construction problems with three given lengths such as sides, medians, bisectors, or others. We give three examples admitting elegant constructions. [4]

7.1. *Construction from a sidelength and the corresponding median and angle bisector.* Given the length $2a$ of a side of a triangle, and the lengths m and t of the median and the angle bisector on the same side, to construct the triangle. This is Problem 1054(a) of the *Mathematics Magazine* [6]. In his solution, Howard Eves denotes by z the distance between the midpoint and the foot of the angle bisector on the side $2a$, and obtains the equation

$$z^4 - (m^2 + t^2 + a^2)z^2 + a^2(m^2 - t^2) = 0,$$

from which he concludes constructibility (by ruler and compass). We devise a simple construction, assuming the data given in the form of a triangle $AM'T$ with $AT = t$, $AM' = m$ and $M'T = a$. See Figure 33. Writing $a^2 = m^2 + t^2 - 2tu$, and $z^2 = m^2 + t^2 - 2tw$, we simplify the above equation into

$$w(w - u) = \frac{1}{2}a^2. \tag{6}$$

[4] Construction 3 (Figure 7) solves the construction problem of triangle ABC given angle A, side a, and the length t of the bisector of angle A. See Footnote 3.

Note that u is length of the projection of AM' on the line AT, and w is the length of the median AM on the bisector AT of the sought triangle ABC. The length w can be easily constructed, from this it is easy to complete the triangle ABC.

Construction 22. (1) *On the perpendicular to AM' at M', choose a point Q such that $M'Q = \frac{M'T}{\sqrt{2}} = \frac{a}{\sqrt{2}}$.*

(2) *Construct the circle with center the midpoint of AM' to pass through Q and to intersect the line AT at W so that T and W are on the same side of A.* (*The length w of AW satisfies* (6) *above*).

(3) *Construct the perpendicular at W to AW to intersect the circle $A(M')$ at M.*

(4) *Construct the circle $M(a)$ to intersect the line MT at two points B and C.*

The triangle ABC has AT as bisector of angle A.

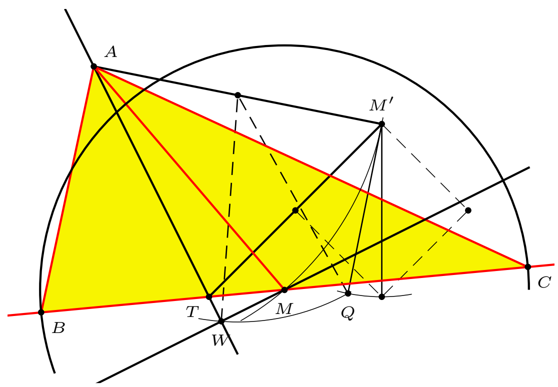

Figure 33

7.2. *Construction from an angle and the corresponding median and angle bisector.* This is Problem 1054(b) of the *Mathematics Magazine*. See [6]. It also appeared earlier as Problem E1375 of the *American Mathematical Monthly*. See [10]. We give a construction based on Thébault's solution.

Suppose the data are given in the form of a right triangle OAM, where $\angle AOM = A$ or $180^\circ - A$, $\angle M = 90^\circ$, $AM = m$, along with a point T on AM such that $AT = t$. See Figure 34.

Construction 23. (1) *Construct the circle $O(A)$. Let A' be the mirror image of A in M. Construct the diameter XY perpendicular to AA' X the point for which $\angle AXA' = A$.*

(2) *On the segment $A'X$ choose a point P such that $A'P = \frac{t}{2}$. and construct the parallel through P to XY to intersect $A'Y$ at Q.*

(3) *Extend XQ to K such that $QK = QA'$.*

(4) *Construct a point B on $O(A)$ such that $XB = XK$, and its mirror image C in M.*

Triangle ABC has given angle A, median m and bisector t on the side BC.

7.3. *Construction from incenter and feet of altitude and angle bisector on a side.* Given the incenter I, the midpoint D of BC, and the projections X of A on BC, to construct the triangle. This is Problem 1149(c) of the *Mathematics Magazine*. See [15]. The points B and C lie on the line ℓ joining D and X. Let P be the projection of I on ℓ. Then, $I(P)$

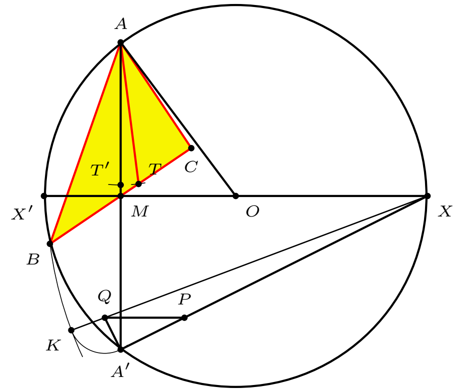

Figure 34

is the incircle of the triangle. Let D and X are on opposite sides of P. If $DX = w$ and $PX = v$, then $w \geq v$ and

$$\left(\frac{a}{2}\right)^2 = v^2 + \frac{w+v}{w-v}r^2.$$

Construction 24. *Construct*
(i) *the mirror image X' of X in P, and the perpendicular to IX' at I to intersect ℓ at Y,*
(ii) *the mirror image Z of X in D,*
(iii) *the circle with diameter YZ to intersect the line IP at Z,*
(iv) *the circle $D(Z)$ to intersect ℓ at B and C.*

These are the vertices on ℓ. The tangents to $I(P)$ from these points *intersect at A.* See Figure 35.

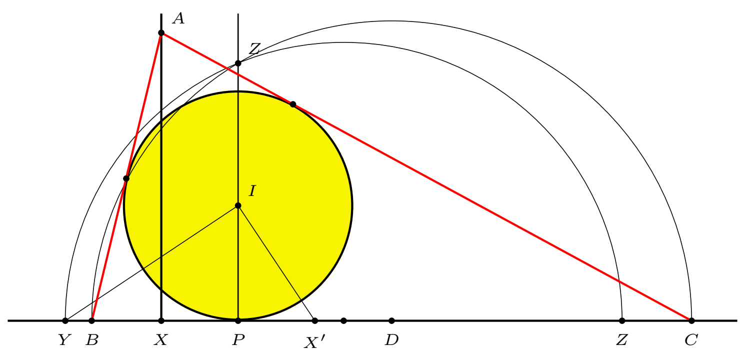

Figure 35

References

[1] G. L. Alexanderson, A conversation with Leon Bankoff, *College Math. Journal*, 23 (1992) 98–117.
[2] J.-L. Ayme, Sawayama and Thébault's theorem, *Forum Geom.*, 3 (2003) 225–229.
[3] L. Bankoff, Are the twin circles of Archimedes really twin ?, *Math. Mag.* 47 (1974) 214–218.
[4] L. Bankoff, A mixtilinear adventure, *Crux Math.*, 9 (1983) 2–7.
[5] M. Beger, *Geometry II*, Springer-Verlag, 1987.
[6] J. C. Cherry and H. Eves, Problem 1054, *Math. Mag.*, 51 (1978) 305; solution, 53 (1980) 52–53.
[7] H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965.

[8] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.
[9] H. Fukagawa and J. F. Rigby, *Traditional Japanese Mathematics Problems of the 18th and 19th Centuries*, SCT Press, Singapore, 2002.
[10] L. D. Goldstone, V. Thébault and R. Woods, Problem E 1375, *Amer. Math. Monthly*, 66 (1959) 513; solution, 67 (1960) 185–186.
[11] T. L. Heath, *The Works of Archimedes*, 1912, Dover reprint.
[12] L. Lopes, *Manuel de Cosntruction de Triangles*, QED Texte, Québec, 1996.
[13] V. Thébault, Problem 3887, *Amer. Math. Monthly*, 45 (1938) 482–483.
[14] W. Wernick, Triangle constructions with three located points, *Math. Mag.*, 55 (1982) 227–230.
[15] W. Wernick and J. M. Stark, Problem 1149, *Math. Mag.*, 55 (1982) 236; solution, 57 (1984) 43–45.
[16] P. Woo, Simple constructions of the incircle of an arbelos, *Forum Geom.*, 1 (2001) 133–136.
[17] P. Yiu, Mixtilinear incircles, *Amer. Math. Monthly*, 106 (1999) 952–955.

Unit: Geometric constructions. Lessons. Constructing bisectors of lines and angles. Learn. Geometric constructions: perpendicular bisector.Â Constructing regular polygons inscribed in circles. Learn. Geometric constructions: circle-inscribed square. This is an interactive course on geometric constructions, a fascinating topic that has been ignored by the mainstream mathematics education. It is all about drawing geometric figures using specific drawing tools like straightedge, compass and so on. This classical topic in geometry is important because. the foundation of geometry is mostly inspired by what we can do with all these drawing tools, and. it involves a lot of beautiful mathematics that shows the interplay between geometry and algebra. â€¢ Construction of primitive geometric forms (points, lines and planes etc.) that serve as the building blocks for more complicated geometric shapes. â€¢ Defining the position of the object in space. Lines and Planes. Solids. Curved surfaces. Primitive geometric forms. â€¢ Point â€¢ Line â€¢ Plane â€¢ Solid â€¢ â€¦â€¦etc. The basic 2-D geometric primitives, from which other more complex geometric forms are derived. Points, Lines, Circles, and Arcs. Point. Compass-and-straightedge geometric constructions are familiar to most students from high-school geometry. Nowadays, they are viewed by most as a quaint curiosity of no more than academic interest. To the ancient Greeks and Egyptians, however, geometric constructions were useful tools, and for some, everyday tools, used for construction and surveying, among other activities.Â Lang, Origami and Geometric Constructions points where a crease hits a folded edge. Construction of Rectangle Given the Sides. To construct quadrilaterals when four sides and one angle is given H. Introduction to Geometric Constructions. As you are familiar with various shapes, you can draw them with your hands. You are well aware with the geometric constructions of a line segment of a certain measurement, a square, a rectangle or a triangle with the help of a ruler. In this section, we are going to learn some more geometric constructions with the help of a compass, a ruler, and a protector.